NOTES ON
ANALYTICAL GEOMETRY

AN APPENDIX

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PREFACE

The present work is intended for students who have already covered a considerable portion of the usual course in analytical geometry, and is written rather by way of appendix to the ordinary text-book, than as an attempt to impart the rudiments of the subject. The use of the single variable is developed at some length, and the elementary theory of equations is applied comprehensively to problems in analytical conics. On these methods depend a large proportion of the problems in analytical geometry, though comparatively little space is afforded them (no doubt with wisdom in the case of beginners) in most text-books. If thoroughly understood, however, a method which can be so constantly applied will furnish one of those plain high-roads which are so necessary to the student, if he is to get a real grasp of the subject and readiness in applying what he knows. The use of the equation of the straight line which passes through a given point and has a given direction is enlarged upon also, and various well-known equations are reduced conveniently to this form. Its applications are many, and in particular it often provides a linear expression for a quantity where otherwise we should be troubled with a quadratic surd.

Moreover, though only elementary work is done, and simple results arrived at, it is hoped that the analysis is arranged and developed in such manner as to give the student some workable notions of analytic research, which may perhaps assist him in higher and harder problems. To this end several results are given, which the author believes to be new, though perhaps they are not of great importance or of very wide application. The reader is strongly recommended to work out the examples given in the text *before* reading the solutions, so that he may discover the difficulties they are intended to illustrate. My thanks are due to the Delegates of the Clarendon Press for permission to copy many of the appended exercises from the various University examination papers. I would also acknowledge my indebtedness to my former tutor, Mr. C. H. Sampson, M.A., of Brasenose College, Oxford, who has given me encouragement in this work from time to time; the extent of my debt to him in general it is impossible to estimate. Finally, my best thanks are due to my former pupil, Mr. Charles W. Adams (of Westminster and Trinity College, Cambridge), who has corrected the proofs, made many valuable suggestions, and compiled the index.

A. C. J.

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CHAPTER I

THE STRAIGHT LINE

§ **1.** The general equation in Cartesian coordinates of a straight line is

$$Ax+By+C=0.$$

When A, B, and C are constant quantities this is the equation satisfied by the coordinates (x, y) of any point on the line. On the other hand, if A, B, C are regarded as variable constants, it is an equation satisfied by the constants A, B, and C, corresponding to any straight line which passes through the point (x, y), this point being fixed. The most important forms to which this equation of a straight line may be reduced are as follows:—

$$\frac{x}{a}+\frac{y}{b}=1. \quad \ldots \quad \ldots \quad \text{(I)}$$

The constants a and b are the lengths of the intercepts made by the straight line on the coordinate axes. This equation may also be regarded as one connecting the lengths of the intercepts made on the coordinate axes by a variable straight line which passes through the point (x, y); in this sense (a, b) may be regarded as the coordinates of such lines.

$$x\cos\alpha+y\sin\alpha=p. \quad \ldots \quad \ldots \quad \text{(II)}$$

In this form of the equation p is the length of the perpendicular drawn from the origin to the straight line, and α the inclination of this perpendicular to the axis of x, i. e. the angle α determines the direction of the line.

$$\frac{x-a}{\cos\theta}=\frac{y-b}{\sin\theta}=r. \quad \ldots \quad \ldots \quad \text{(III)}$$

In these equations (a, b) is a fixed point on the straight

line; (x, y) any point on the line; r the distance between the points (a, b) and (x, y); and θ the inclination of the straight line to the axis of x.

The two constants which occur in each of these equations correspond to the two conditions which are necessary to determine a straight line; e. g.

(i) Two points. (ii) One point and the direction.

(iii) Two circles of which it is the radical axis.

The last equation (III) appears to contain three constants; two of these however, a and b, are not independent, being connected by virtue of the fact that the point (a, b) is on the straight line. The equation reduces to

$$x \sin\theta - y\cos\theta = a\sin\theta - b\cos\theta = p.$$

The general equation of a straight line may be put in this form in the following manner:—

Choose any point (a, b) on the straight line, any value a being substituted for x and the corresponding value b of y determined. Thus

$$Ax + By + C = 0$$

and

$$Aa + Bb + C = 0,$$

hence, eliminating C, $\dfrac{x-a}{-B} = \dfrac{y-b}{A}$;

therefore if θ is taken so that $\tan\theta = -\dfrac{A}{B}$, it follows that

$$\frac{x-a}{\cos\theta} = \frac{y-b}{\sin\theta} = \left\{\frac{(x-a)^2+(y-b)^2}{\cos^2\theta+\sin^2\theta}\right\}^{\frac{1}{2}} = r.$$

Further, if the quantities a, b, and θ are known, since

$$x = r\cos\theta + a,$$
$$y = r\sin\theta + b,$$

the coordinates of any point on the straight line can be expressed in terms of the single variable r; i. e. the point $(r\cos\theta + a,\ r\sin\theta + b)$ lies on the straight line for all values of the variable r.

§ 2. The following examples illustrate the method of using equation (III).

(i) *To find the coordinates of a point on the straight line*

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1,$$

whose distance from the point $(a\cos\theta,\ b\sin\theta)$ *is*

$$\{a^2\sin^2\theta+b^2\cos^2\theta\}^{\frac{1}{2}}.$$

The equation of the line may be written

$$\frac{x-a\cos\theta}{-a\sin\theta}=\frac{y-b\sin\theta}{b\cos\theta}=\frac{r}{\{a^2\sin^2\theta+b^2\cos^2\theta\}^{\frac{1}{2}}};$$

hence, substituting for r the given value, the values of the coordinates (x, y) of the required point are

$$a(\cos\theta-\sin\theta),\quad b(\cos\theta+\sin\theta).$$

(ii) *To find the distances from a point* (a, b) *of the intersections of a straight line through the point in the direction* θ *with the circle* $x^2+y^2=d^2$.

Any point on the line is $(r\cos\theta+a,\ r\sin\theta+b)$, this point lies on the given circle if

$$(r\cos\theta+a)^2+(r\sin\theta+b)^2=d^2,$$

i. e. if $\quad r^2+2r(a\cos\theta+b\sin\theta)+a^2+b^2-d^2=0.$

This is a quadratic equation in r, giving the required lengths.

Incidentally it may be noticed that:

(a) Since the equation is quadratic, any straight line meets the circle in two points.

(b) If the values of r given by this equation are r_1 and r_2, then

$$r_1 r_2=a^2+b^2-d^2=\text{a constant}.$$

This result corresponds to Euc. III. 35, 36.

§ 3. Several well-known results may be easily deduced from the equation of a straight line in the form (III); they are given here in order to familiarise the reader with the meaning of the various elements in this equation.

(i) *To find the condition that two straight lines should be parallel.*

If $Ax+By+C=0$ is one of the straight lines, it has been shown that if it is put into the form (III),

$$\tan\theta = -\frac{A}{B}.$$

The element θ determines the direction of the straight line and is therefore the same in the equations of all parallel straight lines. Hence, if two straight lines are parallel, the ratio $\frac{A}{B}$ of the coefficients of x and y in their respective equations is the same. It follows that in general $Ax+By+C=0$ represents a system of parallel straight lines for different values of C.

(ii) *To find the equation of a straight line through the point (a, b) perpendicular to the straight line $Ax+By+C=0$.*

If θ is the direction of the given line, so that $\tan\theta = -\frac{A}{B}$, then $\theta \pm \frac{\pi}{2}$ is the direction of the required line.

The equation of the line through the point (a, b) in the direction $\theta \pm \frac{\pi}{2}$ is $\frac{x-a}{-\sin\theta} = \frac{y-b}{\cos\theta}$; hence the required equation is

$$B(x-a)-A(y-b)=0.$$

(iii) *To find the angle between two straight lines whose equations are* $Ax+By+C=0$ *and* $A'x+B'y+C'=0$.

If θ and ϕ are the directions of these lines respectively, the angle between them is $(\theta-\phi)$, which in terms of the coefficients of the given equations is

$$\tan^{-1}\frac{A'B-AB'}{AA'+BB'}.$$

(iv) *To find the perpendicular distance of the point (a, b) from the straight line* $Ax+By+C=0$.

The equation of a straight line through the point (a, b), perpendicular to the given line, is

$$\frac{x-a}{A}=\frac{y-b}{B}=\pm\frac{r}{\sqrt{A^2+B^2}}.$$

This line meets the given line at a point whose distance (r) from the point (a, b) is given by

$$A\left(\pm\frac{rA}{\sqrt{A^2+B^2}}+a\right)+B\left(\pm\frac{rB}{\sqrt{A^2+B^2}}+b\right)+C=0.$$

Hence $$r=\pm\frac{Aa+Bb+C}{\sqrt{A^2+B^2}}.$$

§ 4. The form of the equation of a straight line given in § 1 (III) may be usefully applied to an analytical discussion of problems in the theory of inversion. The following examples will sufficiently illustrate the method:—

(i) *To find the inverse of a straight line with regard to any point.*

Let the origin of coordinates (O) be the centre of inversion, and let the equation of the straight line be

$$Ax+By+C=0.$$

The equation of any straight line through the origin is

$$\frac{x}{\cos\theta}=\frac{y}{\sin\theta}=r.$$

The distance from O of the point (P), in which this line intersects the given line, is given by the equation

$$r(A\cos\theta+B\sin\theta)+C=0.$$

Suppose P' is the point on the inverse corresponding to P and let $OP'=r'$, then $$rr'=k^2.$$

Substituting for r it follows that

$$k^2(A\cos\theta+B\sin\theta)+r'C=0.$$

Now since the point P' is on the straight line OP, its coordinates are $r'\cos\theta$, $r'\sin\theta$; hence the equation of the inverse is

$$k^2(Ax+By)+C(x^2+y^2)=0.$$

This equation represents a circle passing through the centre of inversion, whose centre lies on the line $Bx-Ay=0$ drawn through the centre of inversion perpendicular to the given line.

(ii) *To find the equation of the inverse of a parabola, the vertex being the centre of inversion.*

Let $y^2-4ax=0$ be the equation of the parabola.

Any straight line through the centre of inversion (O), whose equation is

$$\frac{x}{\cos\theta}=\frac{y}{\sin\theta}=r,$$

meets the parabola in points (P), whose distances from O are given by the equation,
$$r^2\sin^2\theta-4ar\cos\theta=0.$$

If r' is the distance from O of the point corresponding to P, since $rr'=k^2$ it follows that

$$k^2\sin^2\theta-4ar'\cos\theta=0.$$

Hence the equation of the inverse is

$$k^2y^2-4ax(x^2+y^2)=0,$$

which may be written
$$x^3=y^2(c-x),$$
where
$$k^2=4ac.$$

(iii) *To prove analytically that the inverse of a system of coaxial circles with respect to a limiting point is a system of concentric circles.*

Let $x^2+y^2+2gx+d^2=0$ be any one of the system of coaxial circles, whose limiting points are $(\pm d,\ 0)$. Any line through a limiting point is
$$\frac{x\mp d}{\cos\theta}=\frac{y}{\sin\theta}=r.$$

This line meets the circle at points whose distances from the centre of inversion are given by the quadratic equation

$$(r\cos\theta\pm d)^2+r^2\sin^2\theta+2g(r\cos\theta\pm d)+d^2=0,$$

i.e.
$$r^2+2r\cos\theta(g\pm d)+2d(d\pm g)=0.$$

The equation connecting the distances of corresponding points on the inverse from the centre of inversion is accordingly

$$\frac{k^4}{(g\pm d)}+2k^2r'\cos\theta\pm 2dr'^2=0.$$

Hence the equation of the inverse curve is

$$\frac{k^4}{(g\pm d)}+2k^2(x\mp d)\pm 2d\left\{(x\mp d)^2+y^2\right\}=0,$$

which represents for various values of g a system of circles whose centre is the fixed point

$$\left\{\pm\frac{2d^2-k^2}{2d},\ 0\right\}.$$

§ **5.** The equation
$$Ax+By+C=0 \qquad \ldots \qquad \ldots \qquad \text{(I)}$$

is usually regarded as that of the locus of a variable point (x, y), whose coordinates satisfy the condition expressed by this equation; it can also be regarded in a converse light. Consider the two constants implied by the ratios $A : B : C$ as variables, and the point (x, y) as fixed. The values of these ratios which satisfy the condition $Ax + By + C = 0$ give the equations of all lines which pass through the fixed point (x, y); the number of such lines is infinite. If, however, some condition, such as

$$f\left(\frac{A}{C}, \frac{B}{C}\right) = 0, \quad \ldots \quad \ldots \quad \text{(II)}$$

be given, the number of such straight lines which pass through a fixed point is reduced to a finite quantity depending upon the degree of equation (II); further, one of the ratios $\left(\frac{B}{C}\right)$ can be theoretically obtained in terms of the other $\left(\frac{A}{C}\right)$, or, in other words, these ratios are functions of a single variable. If any given form of the equation of a straight line, in which the constants are functions of a single variable, is under consideration, this variable represents the line and when known fixes it.

As an example consider the line

$$Ax + By = a^2,$$

where the constants are connected by the relation

$$A^2 + B^2 = a^2.$$

Put $A = a \cos\alpha$, $B = a \sin\alpha$, which satisfies the given condition for all values of α, and includes all possible values of A and B, since each must be less than a; the equation may then be written $x \cos\alpha + y \sin\alpha = a$.

The element α may now be regarded as the coordinate of the straight lines, whose equations are of this form.

A particular form of the equation of a straight line corresponds in general to some geometrical property; in the example above, the line must touch the circle

$$x^2 + y^2 = a^2.$$

The number of such lines which pass through any fixed point (h, k) is limited, for α must satisfy the equation

$$h\cos\alpha + k\sin\alpha = a,$$

i. e. $$(a+h)\tan^2\frac{\alpha}{2} - 2k\tan\frac{\alpha}{2} + a - h = 0.$$

This equation is quadratic in $\tan\frac{\alpha}{2}$, hence two such straight lines pass through any point (h, k), which are real, coincident, or imaginary, according as the roots of this equation in $\tan\frac{\alpha}{2}$ are real, coincident, or imaginary; i. e. according as $h^2+k^2 >=$ or $< a^2$.

This result may be interpreted geometrically thus: through any point two tangents can be drawn to a circle, which are real, coincident, or imaginary, according as the point lies outside, on, or inside the circle.

As another example, consider the system of straight lines whose equations are of the form

$$\lambda^3 x + y + a\lambda = 0.$$

The values of λ corresponding to those straight lines of the system which pass through any fixed point (h, k) are given by the equation $h\lambda^3 + a\lambda + k = 0$;
this equation being cubic, it follows that three such lines pass through any point. If λ_1, λ_2, λ_3 are the three values of λ given by this equation, since the coefficient of λ^2 is zero,

$$\lambda_1 + \lambda_2 + \lambda_3 = 0,$$

which is a condition independent of the coordinates (h, k) of the chosen point.

Hence if the three straight lines

$$x\lambda_1^3 + y + a\lambda_1 = 0$$
$$x\lambda_2^3 + y + a\lambda_2 = 0$$
$$x\lambda_3^3 + y + a\lambda_3 = 0$$

are concurrent, the sum of λ_1, λ_2, and λ_3 must be zero.

This condition is sufficient, as well as necessary, since, provided the three values of λ satisfy this relation, the point

(h, k) can be determined from the relations

$$\frac{a}{h} = \Sigma\lambda_1\lambda_2; \quad \frac{k}{h} = -\lambda_1\lambda_2\lambda_3.$$

(i) *To find the condition that the lines*

$$x\sin 3\alpha + y\sin\alpha = a$$
$$x\sin 3\beta + y\sin\beta = a$$
$$x\sin 3\gamma + y\sin\gamma = a$$

should be concurrent.

If the point (x, y) lies on each of these lines, α, β, and γ must each satisfy the equation $\quad x\sin 3\theta + y\sin\theta = a.$

This may be written

$$4x\sin^3\theta + (y - 3x)\sin\theta - a = 0,$$

which is a cubic equation in $\sin\theta$, therefore α, β, γ are the values of θ given by this equation.

Since the coefficient of $\sin^2\theta$ is zero,

$$\sin\alpha + \sin\beta + \sin\gamma = 0,$$

which is the required condition; for since any two lines meet in a point only one condition is necessary in order that a third should pass through the same point.

(ii) *If the four straight lines*

$$x\sin 2\alpha + y = a\sin(\alpha + \phi)$$
$$x\sin 2\beta + y = a\sin(\beta + \phi)$$
$$x\sin 2\gamma + y = a\sin(\gamma + \phi)$$
$$x\sin 2\delta + y = a\sin(\delta + \phi)$$

meet in a point, prove that

$$\alpha + \beta + \gamma + \delta = n\pi.$$

As in the last example, α, β, γ, δ each satisfy the equation

$$x\sin 2\theta + y = a\sin(\theta + \phi),$$

which reduces to

$$(2x\tan\theta + y + y\tan^2\theta)^2 = a^2(1 + \tan^2\theta)(\tan\theta\cos\phi + \sin\phi)^2.$$

The coefficients of $\tan\theta$ and $\tan^3\theta$ in this equation are equal, hence

$$\Sigma\tan\alpha = \Sigma\tan\alpha\tan\beta\tan\gamma.$$

Hence $\qquad \tan(\alpha + \beta + \gamma + \delta) = 0,$

or, $\qquad \alpha + \beta + \gamma + \delta = n\pi.$

This condition is necessary, but not sufficient, two conditions being required in order that *four* lines should be concurrent.

CHAPTER II

THE RELATION OF THE STRAIGHT LINE TO THE GENERAL CURVE OF THE SECOND DEGREE

§ 6. The general equation of the second degree is usually written

$$\phi(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

A further usual and convenient notation represents by the capital letters A, B, C, F, G, H the minors corresponding to a, b, c, f, g, h, respectively, in the determinant

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

i. e.

$$A \equiv bc - f^2, \qquad F \equiv gh - af.$$
$$B \equiv ca - g^2, \qquad G \equiv hf - bg.$$
$$C \equiv ab - h^2, \qquad H \equiv fg - ch.$$

It follows algebraically that

$$\Delta a = BC - F^2, \text{ \&c.,}$$

and

$$\Delta f = GH - AF, \text{ \&c.}$$

§ 7. In this chapter the general properties of the locus represented by the general equation of the second degree

$$\phi(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{(I)}$$

are deduced by considering its relation to the straight line

$$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta} = r. \quad \ldots \quad \ldots \qquad \text{(II)}$$

It should be again noted that in using this equation (II) we have three elements at our disposal (viz. the point (α, β), the angle θ, and the length r) any one of which

may be considered variable; thus the properties can be determined:—

(1) Of points which are on a straight line drawn in the direction θ, which are a variable distance r from the point (α, β) determined from some further geometrical condition.

(2) Of points which lie on a pencil of straight lines drawn through the fixed point (α, β) in a variable direction, which are at a determinate distance r from the point (α, β).

(3) Of points (α, β) which lie on a system of parallel straight lines drawn in a constant direction θ, which have some further geometrical property in virtue of their distances r from other determinate points on the straight lines.

§ 8. The coordinates of any point on the straight line § 7 (II) are given by $x = r\cos\theta + \alpha$, $y = r\sin\theta + \beta$; the value of r or θ determines this point according as r or θ is considered variable.

The point (x, y) will also lie on the locus

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

if the substitution of these values of x and y in this equation satisfies it; hence the equation

$$\begin{aligned} r^2 (a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) \\ + 2r\{(a\alpha + h\beta + g)\cos\theta + (h\alpha + b\beta + f)\sin\theta\} \\ + \phi(\alpha, \beta) = 0 \qquad \text{(III)} \end{aligned}$$

gives:

(1) The distances (r) of points which lie on the straight line and the locus from the point (α, β), i.e. the distances of the intersections of the straight line and the curve of the second degree from the point (α, β);

or, (2) if θ is considered variable it gives the directions (θ) of straight lines drawn through the point (α, β), whose intersections with the curve of the second degree are distant r from the point (α, β);

or, (3) if (α, β) is considered variable it gives the locus of points (α, β), which are distant a length r from points on

the curve of the second degree measured in a fixed direction θ.

§ **9.** Since the equation § 8 (III) is quadratic in r, it is clear that to a given value of θ there correspond two values of r; in other words, any straight line through the point (α, β) meets the curve represented by the general equation of the second degree in two real, coincident, or imaginary points, according as the roots of the equation § 8 (III) are real, coincident, or imaginary. Since (α, β) is any point, it follows that *every straight line meets the curve of the second degree in two points.*

Again, if r is regarded as constant, the equation § 8 (III) is biquadratic in $\tan\theta$; hence four straight lines can be drawn through the point (α, β), whose intersections with the given locus are distant a length r from the point (α, β); in other words, a circle of radius r and centre (α, β) cuts the curve in four points which, since imaginary roots occur in pairs, must be all real, two real and two imaginary, or all imaginary. The radius r and the coordinates of the centre (α, β) can have any values; hence every circle meets the curve of the second degree in four points.

§ **10.** *To find the centre of the conic represented by the general equation of the second degree.*

If any chord of this conic is bisected at the point (α, β) the values of r obtained from the equation § 8 (III) must be equal in magnitude and opposite in sign; in this case the coefficient of r must be zero, hence

$$(a\alpha+h\beta+g)\cos\theta+(h\alpha+b\beta+f)\sin\theta = 0,$$

which gives $$\tan\theta = -\frac{a\alpha+h\beta+g}{h\alpha+b\beta+f}.$$

Since the tangent of an angle can have any value, a real value of θ can always be found to satisfy this relation. It follows that through any point (α, β) one, and in general

only one, chord of the conic can be drawn bisected at the point (α, β).

Note. The intersections of the chord and conic may not be real, but the middle point of the straight line joining the two imaginary points is real.

In the particular case in which the coordinates of the point (α, β) satisfy both of the equations

$$\left.\begin{aligned} a\alpha+h\beta+g=0 \\ h\alpha+b\beta+f=0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \text{(IV)}$$

the coefficient of r vanishes for all values of θ, hence every chord through the point (α, β) is bisected at this point. This point is called the centre of the conic; its coordinates may be found by solving equations (IV) simultaneously; these coordinates are

$$\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right) \text{ or } \left(\frac{G}{C}, \frac{F}{C}\right).$$

NOTE (1). If the origin is the centre of the conic, then

$$hf=bg \text{ and } af=hg;$$

therefore $f=g=0$ unless $ab-h^2$ is zero. The general equation of a conic referred to coordinate axes through its centre is consequently

$$ax^2+2hxy+by^2+c=0.$$

NOTE (2). If $ab-h^2=0$ the coordinates of the centre are infinite, the general equation then represents a parabola. This condition implies that the terms of the second degree, viz.

$$ax^2+2hxy+by^2,$$

form a perfect square.

NOTE (3). If $F=G=C=0$, the coordinates of the centre become indeterminate, and the curve of the second degree reduces to a pair of parallel straight lines.

NOTE (4). If the point (α, β) is the centre of the curve, the length of the radius drawn from the point (α, β) to the curve is given by

$$r^2=-\frac{\phi(\alpha, \beta)}{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta}.$$

If the length of this radius is constant for all values of θ, the expression $a\cos^2\theta+b\sin^2\theta+2h\sin\theta\cos\theta$ must be constant for all values of θ, consequently

$$a=b \text{ and } h=0.$$

It follows that the general equation of the second degree represents a circle when $a=b$ and $h=0$. The most general equation of a circle is

$$x^2+y^2+2gx+2fy+c=0.$$

§ **11.** *To find the locus of the middle points of parallel chords of the conic.*

If (α, β) is the middle point of one of the parallel chords drawn in the direction θ, it has been shown that

$$(a\alpha+h\beta+g)\cos\theta+(h\alpha+b\beta+f)\sin\theta=0.$$

Hence the locus of the middle points of all chords drawn in the direction θ is

$$(ax+hy+g)\cos\theta+(hx+by+f)\sin\theta=0.$$

This equation represents a straight line passing through the centre of the conic, i.e. a diameter of the conic.

The inclination θ' of this straight line to the axis of x is given by

$$\tan\theta'=-\frac{a+h\tan\theta}{h+b\tan\theta};$$

hence if the middle points of chords drawn in the direction θ lie on the diameter drawn in the direction θ',

$$a+h(\tan\theta+\tan\theta')+b\tan\theta\tan\theta'=0.$$

From the symmetry of this condition it is clear that if a diameter in the direction θ bisects all chords drawn in the direction θ', a diameter in the direction θ' will bisect all chords drawn in the direction θ.

Diameters so related are said to be conjugate; hence if θ, θ' are the directions of any pair of conjugate diameters of the general curve of the second degree,

$$a+h(\tan\theta+\tan\theta')+b\tan\theta\tan\theta'=0.$$

§ **12.** *To find the conditions that the curve represented by the general equation of the second degree should be an ellipse, a parabola, an hyperbola, or a rectangular hyperbola.*

The equation giving the lengths of the segments of a chord drawn through the point (α, β) in the direction θ, has been shown to be

$$r^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)$$
$$+ 2r\{(a\alpha + h\beta + g)\cos\theta + (h\alpha + b\beta + f)\sin\theta\} + \phi(\alpha, \beta) = 0.$$

If the coefficient of r^2 in this equation is zero, i.e. if

$$a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta = 0,$$

one of the segments is infinite in length, and the line drawn through the point (α, β) in the direction θ meets the curve at one point at infinity. Since this equation is quadratic in $\tan\theta$, there can be drawn through any point (α, β) two lines to meet the curve at infinity, whose directions are given by this equation. The equation being independent of the coordinates (α, β), it follows that if one line drawn in the direction θ cuts the curve at infinity, all lines in this direction cut the curve at infinity.

Again, if the coefficient of r also vanishes, i.e. if (α, β) is the centre, both values of r are infinite; hence lines drawn through the centre in directions given by

$$b\tan^2\theta + 2h\tan\theta + a = 0$$

meet the curve in two coincident points at infinity. These two lines are called the asymptotes of the conic and are the pair of tangents drawn from the centre to the conic.

The asymptotes are real, coincident, or imaginary, according as the roots of the equation in $\tan\theta$ are real, coincident, or imaginary, i.e. according as

$$ab - h^2 \text{ is } < = \text{ or } > 0.$$

If they are real, the curve extends to infinity, and is called an hyperbola.

If θ_1, θ_2 are the directions of the asymptotes, they are at right angles if

$$\tan\theta_1 \tan\theta_2 = -1,$$

i.e. $$a+b = 0,$$

which is the condition that the general equation should represent a rectangular hyperbola.

If the asymptotes are imaginary, the curve does not extend to infinity; it is accordingly a closed curve, and is called an ellipse.

If the asymptotes are coincident $\quad ab-h^2 = 0,$

which is also the condition that the centre should lie at infinity; hence in this case the line at infinity touches the curve, which is called a parabola.

§ 13. *To find the equation of the asymptotes.*

The directions of the asymptotes are given (§ 12) by the equation

$$a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta = 0,$$

and any point on the corresponding lines satisfies the equation

$$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta}$$

if the point (α, β) is the centre.

Hence, substituting for $\cos\theta$ and $\sin\theta$, the equation to the asymptotes becomes

$$a(x-\alpha)^2 + 2h(x-\alpha)(y-\beta) + b(y-\beta)^2 = 0,$$

which reduces to

$$ax^2 + 2hxy + by^2 - 2x(a\alpha + h\beta) - 2y(h\alpha + b\beta) \\ + a\alpha^2 + b\beta^2 + 2h\alpha\beta = 0.$$

Since (α, β) is the centre,

$$a\alpha + h\beta + g = 0$$
$$h\alpha + b\beta + f = 0,$$

hence
$$
\begin{aligned}
a\alpha^2+2h\alpha\beta+b\beta^2-c
&=\alpha(a\alpha+h\beta)+\beta(h\alpha+b\beta)-c\\
&=-g\alpha-f\beta-c\\
&=-\frac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2}\\
&=-\frac{\Delta}{C}.
\end{aligned}
$$

Hence the equation of the asymptotes of the conic
$$\phi(x, y)=0$$
is
$$\phi(x, y)-\frac{\Delta}{C}=0$$
which differs from the equation of the conic only by the constant $\dfrac{\Delta}{C}$.

NOTE (1). Since the equation
$$\phi(x, y)-\frac{\Delta}{C}=0$$
represents a pair of straight lines, it is clear that, if $\Delta=0$ identically, $\phi(x, y)=0$ must represent a pair of straight lines. Consequently, the condition that the general equation of the second degree should represent a pair of straight lines is $\Delta=0$.

NOTE (2). If $\Delta=0$ and $C=0$, the ratio $\dfrac{\Delta}{C}$ becomes indeterminate. It may be shown that under these conditions the locus $\phi(x, y)=0$ reduces to a pair of parallel straight lines, viz. $$(hx+by+f)^2+c-f^2=0.$$

§ **14.** *To find the direction and magnitude of the axes of the conic represented by the general equation of the second degree.*

Def.:—An axis is a diameter of a conic whose length is a maximum or minimum.

If the point (α, β) is the centre of the conic, the length of a radius through the centre in the direction θ is given by

$$r^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta) + \phi(\alpha, \beta) = 0. \qquad \text{(I)}$$

The value of r is a maximum or minimum when

$$a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$$

is a minimum or maximum.

This expression may be written

$$\tfrac{1}{2}\{(a+b) + (a-b)\cos 2\theta + 2h\sin 2\theta\}.$$

But $\quad \{(a-b)\cos 2\theta + 2h\sin 2\theta\}^2$

$$= (a-b)^2 + 4h^2 - \{(a-b)\sin 2\theta - 2h\cos 2\theta\}^2.$$

The maximum and minimum values of the first expression are accordingly

$$\tfrac{1}{2}\{a + b \pm \sqrt{(a-b)^2 + 4h^2}\},$$

and the corresponding values of θ are given by

$$\tan 2\theta = \frac{2h}{a-b}, \quad \ldots \quad \ldots \quad \text{(II)}$$

which equation consequently gives the directions of the axes. The lengths of the axes can be found by substituting the values of θ given by the equation (II) in the expression found for r; this process is equivalent to eliminating θ between equations (I) and (II). These equations can be written

$$h\tan^2\theta + (a-b)\tan\theta - h = 0$$

$$\left(b + \frac{\phi(\alpha, \beta)}{r^2}\right)\tan^2\theta + 2h\tan\theta + \left(a + \frac{\phi(\alpha, \beta)}{r^2}\right) = 0.$$

Eliminate $\tan\theta$ by cross-multiplication, then

$$\left(a + \frac{\phi(\alpha, \beta)}{r^2}\right)\left(b + \frac{\phi(\alpha, \beta)}{r^2}\right) = h^2,$$

an equation in r^2 whose roots give the lengths of the required semi-axes.

This equation may be more simply derived from the following consideration:—

Since the length of any radius drawn in the direction θ is given by

$$r^2(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)+\phi(\alpha,\beta)=0,$$

or, $$\left(b+\frac{\phi(\alpha,\beta)}{r^2}\right)\tan^2\theta+2h\tan\theta+\left(a+\frac{\phi(\alpha,\beta)}{r^2}\right)=0,$$

which is quadratic in $\tan\theta$, there are in general two radii of any given length r; in the case of the axes these lengths are a maximum or minimum, hence the directions of the pair of equal diameters, whose lengths are equal to the length of one of the axes, become coincident; the condition that the above equation should give coincident values of θ is

$$\left(a+\frac{\phi(\alpha,\beta)}{r^2}\right)\left(b+\frac{\phi(\alpha,\beta)}{r^2}\right)=h^2,$$

which is the required equation.

It follows at once from § 7 that, since the centre lies on the asymptotes,

$$\phi(\alpha,\beta)=\frac{\Delta}{C},$$

and the equation giving the lengths of the semi-axes can be written

$$\left(a+\frac{\Delta}{Cr^2}\right)\left(b+\frac{\Delta}{Cr^2}\right)=h^2. \quad \ldots \quad \ldots \quad \text{(III)}$$

NOTE (1). Since the ratio of the squares of the axes is $1 \backsim e^2$, an equation giving the eccentricity (e) can at once be found.

NOTE (2). The axes are the lines through the centre (α, β) drawn in the directions given by

$$\tan 2\theta=\frac{2h}{a-b},$$

or, $$h\sin^2\theta+(a-b)\sin\theta\cos\theta-h\cos^2\theta=0.$$

The coordinates of any point on a line through the point (α, β) in the direction θ satisfy

$$\frac{x-\alpha}{\cos\theta}=\frac{y-\beta}{\sin\theta},$$

hence the equation of the axes is

$$h(x-\alpha)^2-(a-b)(x-\alpha)(y-\beta)-h(y-\beta)^2=0,$$

which represents a pair of lines passing through the centre and parallel to the lines $h(x^2-y^2)=(a-b)xy$.

Solving this equation for $\frac{x}{y}$, it follows that the equations of these straight lines are

$$hx=\left(a+\frac{\Delta}{Cr_1^2}\right)y\,;\quad hx=\left(a+\frac{\Delta}{Cr_2^2}\right)y$$

respectively, where r_1, r_2 are the semi-axes.

§ **15.** *To find the equation of a pair of tangents drawn to a conic from any point.*

The lengths (r) of the intercepts made by the conic on any straight line drawn from the point (α, β) in the direction θ are given by the equation § 8 (III)

$$\begin{aligned} r^2(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)& \\ +2r\{(a\alpha+h\beta+g)\cos\theta+(h\alpha+b\beta+f)\sin\theta\}& \\ +\phi(\alpha, \beta)=0.& \end{aligned}$$

If this straight line is a tangent to the conic, the two values of r given by this equation are coincident, i. e. the equation has equal roots. The corresponding condition is

$$\begin{aligned} \{(a\alpha+h\beta+g)\cos\theta+(h\alpha+b\beta+f)\sin\theta\}^2& \\ =\phi(\alpha, \beta)\{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta\}.& \quad \text{(I)} \end{aligned}$$

This equation may be regarded as one to find the directions (θ) of the tangents from the point (α, β) to the conic; it is quadratic in $\tan\theta$, hence from any point two tangents can be drawn to a conic, which are real, coincident, or imaginary, according as the roots of this equation in $\tan\theta$ are real, coincident, or imaginary.

Note. The condition that it should be possible to draw from any point two real, coincident, or imaginary tangents is clearly the same as that the point should lie outside, on, or inside the conic.

The equation (I) can be written

$$\begin{aligned} (C\alpha^2-2G\alpha+A)\tan^2\theta-2(C\alpha\beta-F\alpha-G\beta+H)\tan\theta& \\ +(C\beta^2-2F\beta+B)=0.& \end{aligned}$$

The roots of this equation are real, coincident, or imaginary, according as

$$(C\alpha\beta - F\alpha - G\beta + H)^2 \text{ is} > = \text{or} < (C\alpha^2 - 2G\alpha + A)(C\beta^2 - 2F\beta + B),$$

which immediately reduces to

$$\Delta\,\phi(\alpha, \beta) \text{ is} < = \text{or} > 0.$$

Hence a point (α, β) lies outside, on, or inside the conic, according as $\Delta\,\phi(\alpha, \beta)$ is negative, zero, or positive.

Again, since the coordinates of any point on the tangent satisfy the equation

$$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta},$$

it follows that the equation of the pair of tangents is

$$\{(a\alpha+h\beta+g)(x-\alpha)+(h\alpha+b\beta+f)(y-\beta)\}^2 = \phi(\alpha, \beta)\{a(x-\alpha)^2+2h(x-\alpha)(y-\beta)+b(y-\beta)^2\},$$

which is identical with

$$\phi(x, y)\,\phi(\alpha, \beta) = \{(a\alpha+h\beta+g)x+(h\alpha+b\beta+f)y+g\alpha+f\beta+c\}^2.$$

NOTE (1). If (α, β) lies on the conic,

$$\phi(\alpha, \beta) = 0,$$

and the tangents become coincident. Hence the equation of the tangent at any point (α, β) on the conic is

$$(a\alpha+h\beta+g)x+(h\alpha+b\beta+f)y+(g\alpha+f\beta+c) = 0.$$

NOTE (2). The two tangents from the point (α, β) are at right angles, if the directions θ_1, θ_2 given by equation (I) of this section satisfy the condition

$$1+\tan\theta_1\tan\theta_2 = 0,$$

i.e. if $(a\alpha+h\beta+g)^2+(h\alpha+b\beta+f)^2-(a+b)\,\phi(\alpha, \beta) = 0.$

This equation gives the locus of the point (α, β), tangents from which to the conic are at right angles. It may be written, putting x for α, and y for β,

$$C(x^2+y^2)-2Gx-2Fy+A+B = 0.$$

This represents a circle, called the *director circle*, its centre $\left(\frac{G}{C}, \frac{F}{C}\right)$ is also the centre of the conic, and the square of its radius is equal to the sum of the squares of the semi-axes. (§ 14. III.)

If the conic is a parabola, $C = 0$, and the director circle reduces to
$$2Gx + 2Fy - A - B = 0,$$
which represents the directrix of the parabola.

§ **16.** *To find the lengths of the tangents which can be drawn from any point* (α, β) *to the curve of the second degree.*

It has been already proved that these lengths (r) are given by the equation
$$r^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)$$
$$+ 2r(P\cos\theta + Q\sin\theta) + \phi(\alpha, \beta) = 0. \quad \ldots \quad \text{(I)}$$
where $\quad P \equiv a\alpha + h\beta + g,$ and $Q \equiv h\alpha + b\beta + f,$
and the angle θ satisfies the condition
$$(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)\,\phi(\alpha, \beta)$$
$$= (P\cos\theta + Q\sin\theta)^2. \quad \ldots \quad \text{(II)}$$

If p represents the length of either tangent, since the roots of equation (I) are equal,
$$p^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)$$
$$= \phi = \phi(\cos^2\theta + \sin^2\theta) \quad \ldots \quad \text{(III)}$$
where ϕ is written instead of $\phi(\alpha, \beta)$.

Hence $(bp^2 - \phi)\tan^2\theta + 2hp^2\tan\theta + (ap^2 - \phi) = 0. \quad$ (IV)

From equations (II) and (III) it follows that
$$p^2(P\cos\theta + Q\sin\theta)^2 = \phi^2 = \phi^2(\cos^2\theta + \sin^2\theta),$$
or, $\quad (p^2Q^2 - \phi^2)\tan^2\theta + 2PQp^2\tan\theta + p^2P^2 - \phi^2 = 0. \quad$ (V)

The equations (IV) and (V) are each quadratic in $\tan\theta$; the elimination of $\tan\theta$ from these equations gives an equation in p, the coefficients of which contain only the coefficients in the equation of the conic, and the coordinates of the chosen point (α, β); this eliminant is consequently the equation required giving the lengths of the tangents.

The elimination of $\tan\theta$ gives,

$$4p^4\left[h\left(p^2P^2-\phi^2\right)-PQ\left(ap^2-\phi\right)\right]\left[PQ\left(bp^2-\phi\right)-h\left(p^2Q^2-\phi^2\right)\right]$$
$$=\left[\left(ap^2-\phi\right)\left(p^2Q^2-\phi^2\right)-\left(bp^2-\phi\right)\left(p^2P^2-\phi^2\right)\right]^2$$
$$\equiv p^4\left[p^2\left(aQ^2-bP^2\right)+\phi\left(P^2-Q^2+\overline{b-a}\,\phi\right)\right]^2,$$

which is a quadratic equation in p^2, whose roots are the squares of the lengths of the two tangents which can be drawn from the point (α, β) to the conic.

This result may be reduced to the following:

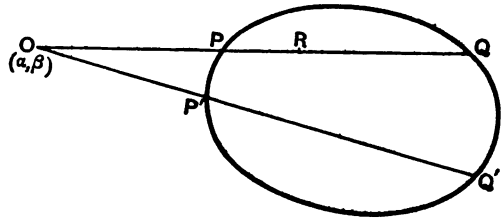
$$p^4\left(L^2+4MN\right)+2p^2\phi\left\{LY+(M+N)X\right\}+\phi^2\left(X^2+Y^2\right)=0,$$

where
$$X\equiv 2\,(h\phi-PQ),$$
$$Y\equiv P^2-Q^2+(b-a)\,\phi,$$
$$L\equiv aQ^2-bP^2,$$
$$M\equiv P\,(aQ-hP),$$
$$N\equiv Q\,(bP-hQ).$$

It is interesting to notice that the lengths of the tangents reduce to zero, not only when $\phi=0$, i.e. when the point (α, β) is on the curve, but also when $X=0$ and $Y=0$ simultaneously. The conditions $X=Y=0$ denote that the point (α, β) is one of the foci of the conic; hence the tangents to a conic from either of the foci, though imaginary, are zero in length.

§ **17.** Let O be the point (α, β) and OPQ the line

$$\frac{x-\alpha}{\cos\theta}=\frac{y-\beta}{\sin\theta}=r.$$



Referring back to equation (III), § 8, it follows that

$$OP+OQ=\frac{-2\left[(a\alpha+h\beta+g)\cos\theta+(h\alpha+b\beta+f)\sin\theta\right]}{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta}$$

and $$OP\,.\,OQ=\frac{\phi(\alpha,\beta)}{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta}.$$

The following propositions may be deduced from these results :—

(1) If the general equation represents a circle it has been shown that $$a=b \text{ and } h=0.$$

In this case $OP\,.\,OQ=\dfrac{\phi(\alpha,\beta)}{a}=$ constant, which corresponds to Euc. III. 35, 36.

(2) If $OP'Q'$ is drawn in the direction θ', the ratio of the rectangles $OP'.OQ'$ and $OP.OQ$ depends only on the directions θ and θ'; the ratio is

$$\frac{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta}{a\cos^2\theta'+2h\sin\theta'\cos\theta'+b\sin^2\theta'}.$$

Since this ratio is independent of the coordinates of the point $O(\alpha,\beta)$, it follows that the ratio of the rectangles contained by the segments of any pair of chords drawn in fixed directions is constant. This ratio is hence also that of:

(i) The squares of the lengths of two tangents parallel to the chords.

(ii) The squares of the parallel diameters.

(3) If straight lines are drawn through two points, $O(\alpha,\beta)$ and $O'(\alpha',\beta')$, in the same direction θ, to meet the curve in the points PQ, $P'Q'$ respectively, the ratio of the rectangles $OP.OQ$ and $O'P'.O'Q'$ is $\dfrac{\phi(\alpha,\beta)}{\phi(\alpha',\beta')}$; hence the ratio of the rectangles contained by the segments of parallel chords through two points is constant and independent of the direction of these chords.

(4) *To find the locus of a point R on the straight line OPQ, when $\{ORPQ\}$ is an harmonic range.*

Let $OR = r$, then

$$\frac{2}{r} = \frac{1}{OP} + \frac{1}{OQ} = \frac{OP+OQ}{OP\,.\,OQ}$$
$$= -\frac{2\,\{(a\alpha+h\beta+g)\cos\theta+(h\alpha+b\beta+f)\sin\theta\}}{\phi(\alpha,\beta)}.$$

Hence

$$r\cos\theta\,(a\alpha+h\beta+g)+r\sin\theta\,(h\alpha+b\beta+f)+\phi(\alpha,\beta)=0.$$

But the coordinates of the point R are given by

$$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta} = r.$$

Hence the locus of R is

$$(x-\alpha)(a\alpha+h\beta+g)+(y-\beta)(h\alpha+b\beta+f)+\phi(\alpha,\beta)=0,$$

i.e. $\quad x(a\alpha+h\beta+g)+y(h\alpha+b\beta+f)+g\alpha+f\beta+c=0.$

This equation represents a straight line, which is called the polar of (α, β) with respect to the conic.

This equation can also be obtained by using another geometrical property of the pole and polar; viz. the polar of a point is the chord of contact of tangents drawn from the pole to the conic.

The distances of the points of contact of the tangents from the point (α, β) are given by the equation,

$$r^2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)=\phi(\alpha,\beta).$$

Hence these points of contact lie on the locus

$$a(x-\alpha)^2+2h(x-\alpha)(y-\beta)+b(y-\beta)^2=\phi(\alpha,\beta),$$

they also lie on the conic $\quad \phi(x,y)=0.$

The former equation thus becomes

$$\begin{aligned}2x(a\alpha+h\beta)+2y(h\alpha+b\beta)&+2(g\alpha+f\beta+c)\\ &= ax^2+2hxy+by^2+c\\ &= -2gx-2fy,\end{aligned}$$

i.e. $\quad x(a\alpha+h\beta+g)+y(h\alpha+b\beta+f)+(g\alpha+f\beta+c)=0.$

(5) ***To find the conditions that two conics should be* (1) *similar,* (2) *similar and similarly situated.***

Definition 1. If the radii drawn from two points respectively, one to each of two conics, in directions which contain a constant angle, are in a constant ratio, the conics are similar.

Definition 2. If the radii in the above definition are drawn in the same direction and their ratio is constant, the conics are similar and similarly situated.

Suppose a straight line is drawn from a point $O\,(\alpha, \beta)$ to a conic meeting it in the points P, Q; and one from the point $O'\,(\alpha', \beta')$ in the same direction to meet another conic in the points P', Q'.

If these two conics are similar and similarly situated, the ratios $\dfrac{OP}{OP'}$ and $\dfrac{OQ}{OQ'}$ are constant, hence also the ratio $\dfrac{OP\,.\,OQ}{OP'\,.\,OQ'}$ is constant.

It will be evident from previous results, that if the conics are
$$\phi\,(x, y) = ax^2 + 2hxy + by^2 + \&c.,$$
and
$$\phi'\,(x, y) = a'x^2 + 2h'xy + b'y^2 + \&c.,$$
then
$$OP\,.\,OQ = \phi\,(\alpha, \beta) \div (a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)$$
$$O'P'\,.\,O'Q' = \phi'\,(\alpha', \beta') \div (a'\cos^2\theta + 2h'\sin\theta\cos\theta + b'\sin^2\theta).$$

The condition above requires that the ratio
$$\frac{a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta}{a'\cos^2\theta + 2h'\sin\theta\cos\theta + b'\sin^2\theta}$$
should be independent of θ.

Hence
$$\frac{a}{a'} = \frac{h}{h'} = \frac{b}{b'}.$$

This is the required condition that the conics should be similar and similarly situated; it is also the condition that their asymptotes should be parallel.

Again, if the straight lines through O and O' are drawn

in the directions θ and ϕ respectively, the ratio of the rectangles $OP.OQ$ and $O'P'.O'Q'$ becomes

$$\frac{a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta}{a'\cos^2\phi+2h'\cos\phi\sin\phi+b'\sin^2\phi}.$$

If the conics are similar, this ratio must be constant when $\theta \sim \phi$ is constant.

Putting $\theta-\phi=\gamma$, the ratio becomes

$$\frac{a\cos^2(\phi+\gamma)+2h\sin(\phi+\gamma)\cos(\phi+\gamma)+b\sin^2(\phi+\gamma)}{a'\cos^2\phi+2h'\sin\phi\cos\phi^2+b'\sin^2\phi},$$

and must be independent of ϕ.

The ratio may be written

$$\frac{a+b+(a-b)\cos 2(\phi+\gamma)+2h\sin 2(\phi+\gamma)}{a'+b'+(a'-b')\cos 2\phi+2h'\sin 2\phi}.$$

Equating the ratios of the coefficients of $\cos 2\phi$, $\sin 2\phi$, and the terms independent of ϕ,

$$\frac{a+b}{a'+b'}=\frac{(a-b)\cos 2\gamma+2h\sin 2\gamma}{a'-b'}=\frac{2h\cos 2\gamma-(a-b)\sin 2\gamma}{2h'}.$$

Hence, eliminating γ,

$$\frac{(a+b)^2}{(a'+b')^2}=\frac{(a+b)^2-(a-b)^2-4h^2}{(a'+b')^2-(a'-b')^2-4h'^2}=\frac{ab-h^2}{a'b'-h'^2}.$$

This is the condition that the conics should be similar, and is also the condition that the angle between the asymptotes of each conic should be the same.

CHAPTER III

THE PARABOLA

§ **18.** The general equation of the second degree represents a parabola when the terms of the second degree form a perfect square; hence the equation of a parabola is of the form

$$(ax+by)^2 = 2gx+2fy+c.$$

The coordinates of any point on this parabola satisfy the equations

$$\left(\frac{ax+by}{\lambda}\right)^2 = \frac{2gx+2fy+c}{\lambda^2} = 1,$$

consequently by solving the equations

$$ax+by = \lambda \text{ and } 2gx+2fy+c = \lambda^2$$

the coordinates of any point on a parabola can always be expressed in terms of a single variable; the value of this variable corresponding to any point is called the parameter of the point.

In the general case such coordinates are too complicated to use with advantage, but in more simple cases their use simplifies the analysis, for the form of the coordinates implies that the corresponding point is on the curve, and hence no separate equation expressing this condition is necessary.

The equation of a parabola takes its simplest form when the axes of coordinates are the axis of the parabola and the tangent at its vertex; the equation then reduces to

$$y^2 = 4ax \quad \dots \quad \dots \quad \dots \quad \text{(I)}$$

in which $4a$ represents the latus-rectum of the parabola.

The coordinates of any point on the parabola represented by equation (I) satisfy the equations

$$\frac{x}{\lambda^2}=\frac{y}{2\lambda}=\frac{a}{1},$$

i.e. $$x = a\lambda^2,\ y = 2a\lambda,$$

for the point $(a\lambda^2, 2a\lambda)$ satisfies equation (I) for all values of λ, and conversely, since the value of λ is not restricted, any point on the parabola can be thus represented; in the following pages the point whose parameter is λ will be referred to as the point λ. With this notation the vertex of the parabola is the point 0.

A similar form for the coordinates of any point on the parabola can be used when any tangent and the diameter through the point of contact are taken as coordinate axes, the equation of the parabola being

$$y^2 = 4a'x,$$

consequently the propositions established in the following sections will be equally true for these axes, provided the proofs do not involve the assumption that the axes are rectangular.

NOTE. If the tangent taken as axis of y makes an angle α with the axis of the parabola, $a' = a \operatorname{cosec}^2 \alpha$.

Example. *To find the coordinates of any point on the parabola whose equation is*

$$x^2 \cos^2 \alpha = 4a(x \cos \alpha + y \sin \alpha)$$

in terms of a single variable.

The equation can be written

$$(x \cos \alpha - 2a)^2 = 4a(y \sin \alpha + a).$$

The coordinates of any point on the curve satisfy the equations

$$\frac{x \cos \alpha - 2a}{2\lambda} = \frac{y \sin \alpha + a}{\lambda^2} = \frac{a}{1}.$$

Hence the coordinates of any point on the curve are

$$x = 2a(\lambda + 1) \sec \alpha;\quad y = a(\lambda^2 - 1) \operatorname{cosec} \alpha.$$

§ **19.** *To find the equation of the tangent at any point on a parabola the coordinates of points on which can be expressed in the form* $x = f_1(\lambda),\ y = f_2(\lambda),$ *where f_1 and f_2 are functions of a single variable.*

The equation of the line joining the two points λ_1, λ_2 is

$$\frac{x - f_1(\lambda_1)}{f_1(\lambda_2) - f_1(\lambda_1)} = \frac{y - f_2(\lambda_1)}{f_2(\lambda_2) - f_2(\lambda_1)}.$$

When $\lambda_1 = \lambda_2$ the denominators on each side of this equation vanish, hence each denominator is divisible by $(\lambda_1 - \lambda_2)$. If this factor is removed and λ_1 is put equal to λ_2 in the result, the equation becomes that of the chord joining two coincident points λ, λ on the parabola, i.e. it is the equation of the tangent to the parabola at the point λ. The resulting equation of the tangent can be written in the notation of the differential calculus,

$$\left(x - f_1(\lambda)\right)\frac{dy}{d\lambda} - \left(y - f_2(\lambda)\right)\frac{dx}{d\lambda} = 0.$$

This method of finding the equation of the tangent is applicable to all curves, the coordinates of points on which can be expressed in terms of a single variable.

Example. *To find the equation of the tangent at any point on the parabola*

$$x^2 \cos^2\alpha = 4a\,(x\cos\alpha + y\sin\alpha).$$

The coordinates of any point on the parabola are

$$x = 2a(\lambda + 1)\sec\alpha,$$
$$y = a(\lambda^2 - 1)\operatorname{cosec}\alpha.$$

The equation of any straight line can be put into the form

$$A\,(y\sin\alpha + a) + B\,(x\cos\alpha - 2a) + Ca = 0.$$

If the points λ, μ are on this line,

$$A\lambda^2 + 2B\lambda + C = 0,$$
$$A\mu^2 + 2B\mu + C = 0.$$

By cross-multiplication

$$\frac{A}{2(\lambda-\mu)}=\frac{B}{\mu^2-\lambda^2}=\frac{C}{2\lambda\mu(\lambda-\mu)},$$

or,

$$\frac{A}{2}=\frac{B}{-(\lambda+\mu)}=\frac{C}{2\lambda\mu}.$$

The equation of the chord joining the points λ, μ is accordingly

$$2(y\sin\alpha+a)-(\lambda+\mu)(x\cos\alpha-2a)+2a\lambda\mu=0.$$

Putting $\mu=\lambda$ in this equation, the equation of the tangent at the point λ becomes

$$(y\sin\alpha+a)-\lambda x\cos\alpha-2a+a\lambda^2=0,$$

or,

$$\lambda x\cos\alpha-y\sin\alpha-a(\lambda+1)^2=0.$$

§ **20.** *To find the points of intersection of the straight line* $Ax+By+a=0$ *and the parabola* $y^2=4ax$.

The coordinates of any point on the parabola are $a\lambda^2$, $2a\lambda$; this point will also lie on the given line if

$$A\lambda^2+2B\lambda+1=0. \quad \ldots \quad \ldots \quad \text{(I)}$$

Since this equation is quadratic in the variable λ any line meets the parabola in two real or two imaginary points; if these points are λ_1, λ_2 then

$$\lambda_1+\lambda_2=-\frac{2B}{A},$$

$$\lambda_1\lambda_2=\frac{1}{A}.$$

Hence the equation of the straight line joining the points λ_1, λ_2 on the parabola, in other words the equation of the chord joining the points λ_1, λ_2, is

$$y(\lambda_1+\lambda_2)-2x=2a\lambda_1\lambda_2.$$

Note (1). The equation of the straight line joining any point λ on the parabola to the vertex is

$$\lambda y-2x=0.$$

Note (2). The direction of the chord depends only on $(\lambda_1+\lambda_2)$, for the inclination θ of the chord to the axis of x is given by

$$2\cot\theta=\lambda_1+\lambda_2.$$

Hence, if $\lambda_1+\lambda_2$ is constant the direction of the chord is constant; and conversely, if the direction of the chords joining pairs of points on the parabola is constant, the parameters of their extremities are connected by the condition
$$\lambda_1+\lambda_2 = \text{constant};$$
or, if y_1, y_2 are the corresponding ordinates,
$$y_1+y_2 = \text{constant}.$$
The y-coordinate of the middle points of such chords is $a(\lambda_1+\lambda_2)$; therefore the y-coordinate of the middle points of parallel chords of a parabola is constant; in other words, the locus of the middle points of a system of parallel chords of a parabola is a straight line parallel to the axis.

This locus is called a diameter; hence all diameters of a parabola are parallel to the axis.

NOTE (3). The parameters of the extremities of all chords which pass through a fixed point (h, k) are connected by the equation
$$(\lambda_1+\lambda_2)k-2h = 2a\lambda_1\lambda_2.$$

Example. *The locus of the middle points of all chords of a parabola which pass through a fixed point is another parabola.*

Let (h, k) be the fixed point, and λ_1, λ_2 the extremities of any chord. The middle point of this chord is given by
$$2x = a(\lambda_1^2+\lambda_2^2) = a(\lambda_1+\lambda_2)^2-2a\lambda_1\lambda_2,$$
$$y = a(\lambda_1+\lambda_2);$$
therefore
$$2a^2\lambda_1\lambda_2 = y^2-2ax.$$
But $\lambda_1\lambda_2$ satisfy the equation
$$(\lambda_1+\lambda_2)k-2h = 2a\lambda_1\lambda_2;$$
hence the equation of the required locus is
$$ky-2ah = y^2-2ax.$$

NOTE (4). The focus of the parabola is the point $(a, 0)$; hence the condition that the chord joining the points λ_1, λ_2 should be a focal chord is
$$\lambda_1\lambda_2+1 = 0.$$

§ **21.** *To find the equation of the tangent to a parabola at any point.*

The tangent at the point λ is the chord joining the two coincident points λ, λ, its equation is consequently given by substituting $\lambda_1 = \lambda_2 = \lambda$ in the equation of a chord.

Therefore the equation of the tangent at the point λ is

$$\lambda y - x = a\lambda^2,$$

or,

$$a\lambda^2 - \lambda y + x = 0.$$

NOTE (1). Any straight line, whose equation is of the form $y = \frac{x}{\lambda} + a\lambda$, touches the parabola $y^2 = 4ax$.

NOTE (2). The geometrical meaning of the parameter λ can be deduced from this equation. If the tangent at the point makes an angle θ with the axis of x, then $\lambda = \cot\theta$.

It is often convenient to use $\cot\theta$ as the variable parameter: the coordinates of any point on the parabola are then $(a\cot^2\theta,\ 2a\cot\theta)$, and the equation of the tangent at this point is

$$y\cot\theta - x = a\cot^2\theta,$$

which may be written in the form

$$\frac{x - a\cot^2\theta}{\cos\theta} = \frac{y - 2a\cot\theta}{\sin\theta} = r.$$

NOTE (3). The equation of the tangent parallel to the chord

$$y(\lambda_1 + \lambda_2) - 2x = 2a\lambda_1\lambda_2$$

is

$$y\left(\frac{\lambda_1 + \lambda_2}{2}\right) - x = a\left(\frac{\lambda_1 + \lambda_2}{2}\right)^2.$$

Hence the point of contact of this tangent is the point $\frac{\lambda_1 + \lambda_2}{2}$; this point is on the diameter bisecting the chord.

It follows that the middle point of any chord, whose equation is of the form $\lambda y - x = k$, lies on the diameter through the point of contact of the parallel tangent

$$\lambda y - x = a\lambda^2,$$

i.e. on the straight line $y = 2a\lambda$.

NOTE (4). *To find the equation of the chord whose middle point is* (h, k).

The tangent to the parabola at the point whose y-coordinate is k, i.e. whose parameter is $\frac{k}{2a}$, is parallel to the required chord. Its equation is

$$\frac{ky}{2a} - x = \frac{ak^2}{4a^2}.$$

Therefore, since the chord passes through the point (h, k), its equation is $$k(y-k) = 2a(x-h).$$

Example. *To find the envelope of the chords of a parabola whose middle points lie on the straight line* $y = mx + c$.

If (h, k) is any point on this line, the equation of the chord whose middle point is (h, k) is $k(y-k) = 2a(x-h)$,

but by hypothesis $$h = mk + c.$$

Hence the equation of the chord may be written

$$k(y-k) = 2a(x-mk-c),$$

or, $$k^2 - k(y+2am) + 2a(x-c) = 0.$$

This equation is quadratic in the variable k; hence the condition that this equation should have equal roots, i.e. that two consecutive lines of the system should meet at the point (x, y), is

$$(y+2am)^2 = 8a(x-c),$$

which is accordingly the equation of the envelope.

§ **22.** The equation of the tangent to a parabola ($y^2 = 4ax$) at the point λ is

$$a\lambda^2 - \lambda y + x = 0. \quad \dots \quad \dots \quad \text{(I)}$$

If λ is variable and (x, y) a fixed point, this equation represents the condition that the tangent at the variable point λ should pass through the point (x, y); hence, the values of λ given by this equation are the parameters of points on the parabola the tangents at which meet at the point (x, y). If λ_1, λ_2 are the roots of this equation, they are the parameters of the points of contact of the tangents which can be drawn from the point (x, y) to the curve.

Note (1). Two tangents can be drawn from any point to a parabola.

Note (2). Since λ_1, λ_2 are the roots of equation (I)

$$x = a\lambda_1\lambda_2\,; \;\; y = a(\lambda_1+\lambda_2),$$

therefore the point of intersection of tangents at the points λ_1, λ_2 is $\{a\lambda_1\lambda_2,\, a(\lambda_1+\lambda_2)\}$, and this point clearly lies on the diameter bisecting the chord joining the points λ_1, λ_2.

Note (3). *To find the polar of the point (h, k) with respect to the parabola.*

If λ_1, λ_2 are the parameters of the points of contact of the tangents from the point (h, k) to the parabola, then

$$h = a\lambda_1\lambda_2\,; \text{ and } k = a(\lambda_1+\lambda_2).$$

The equation of the chord joining the points λ_1, λ_2 is

$$y(\lambda_1+\lambda_2) - 2x = 2a\lambda_1\lambda_2,$$

whence, substituting for $\lambda_1\lambda_2$ in terms of h and k,

$$ky - 2ax = 2ah,$$

which is the required equation.

Note (4). The angle (γ) between the tangents drawn from the point (x, y) to the parabola is given by

$$\tan\gamma = \frac{\lambda_1-\lambda_2}{1+\lambda_1\lambda_2};$$

where λ_1, λ_2 are roots of equation (I).

Hence, $$\tan\gamma = \frac{\sqrt{y^2-4ax}}{x+a} = \frac{\phi^{\frac{1}{2}}}{x+a}.$$

It follows that the locus of points, the tangents from which to the parabola include an angle γ, is the curve

$$(x+a)^2\tan^2\gamma = y^2-4ax.$$

In particular, if the tangents are at right angles, the locus becomes $$x+a=0,$$

which is the equation of the directrix.

Also, if the tangents at the points λ_1, λ_2 are at right angles,

$$\lambda_1 \lambda_2 + 1 = 0,$$

which is identical with the condition that the chord joining the points λ_1, λ_2 should be a focal chord; hence the proposition: 'Tangents at the extremities of a focal chord are at right angles, and their point of intersection lies on the directrix.'

Example (i). *To find the locus of the intersections of tangents at the extremities of chords of a parabola of constant length.*

If λ_1, λ_2 are the parameters of the extremities of a chord, the square of its length

$$= a^2 \left\{(\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2\right\} \left\{(\lambda_1+\lambda_2)^2 + 4\right\} = c^2.$$

Hence, if the intersection of the tangents at λ_1, λ_2 is the point (x, y), since $x = a\lambda_1\lambda_2$, and $y = a(\lambda_1+\lambda_2)$, the equation of the locus of (x, y) is

$$(y^2 - 4ax)(y^2 + 4a^2) = c^2 a^2.$$

Example (ii). *An equilateral triangle circumscribes a parabola; prove that the join of the focus to each vertex of the triangle passes through the point of contact of the opposite side.*

Let the parameters of the points of contact be $\lambda_1, \lambda_2, \lambda_3$.
Since the triangle is equilateral

$$\frac{\lambda_1 - \lambda_3}{1 + \lambda_1\lambda_3} = \frac{\lambda_3 - \lambda_2}{1 + \lambda_2\lambda_3},$$

which may be written $$\frac{\lambda_3^2 - 1}{\lambda_1\lambda_2 - 1} = \frac{2\lambda_3}{\lambda_1 + \lambda_2};$$

this is the condition that the point λ_3 should lie on the line whose equation is

$$\frac{x-a}{\lambda_1\lambda_2 - 1} = \frac{y}{\lambda_1 + \lambda_2},$$

i.e. on the join of the focus to the point $\left\{a\lambda_1\lambda_2,\, a(\lambda_1+\lambda_2)\right\}$.

Example (iii). *To find the locus of a point, the tangents from which to the parabola make equal angles with the line*

$$y = x\cot\theta + c.$$

Let the points of contact of two tangents be λ_1, λ_2.

By hypothesis, $$\frac{\lambda_1 - \tan\theta}{1+\lambda_1\tan\theta} = \frac{\tan\theta - \lambda_2}{1+\lambda_2\tan\theta},$$

i.e. $$(\lambda_1+\lambda_2)(1-\tan^2\theta) = 2\tan\theta\,(1-\lambda_1\lambda_2).$$

If (x, y) is the point of intersection of the tangents, since $x = a\lambda_1\lambda_2$ and $y = a(\lambda_1+\lambda_2)$, the equation of the required locus is

$$y = (a-x)\tan 2\theta.$$

§ **23.** Some properties, peculiar to the parabola, can be discussed, as in the case of the general conic, by considering the relation of a straight line, whose equation is of the form

$$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta} = r,$$

to the parabola, $\phi(x, y) \equiv y^2 - 4ax = 0.$

Substituting for x and y their values in terms of the variable r, the equation of the parabola becomes

$$r^2\sin^2\theta + 2r(\beta\sin\theta - 2a\cos\theta) + \beta^2 - 4a\alpha = 0. \qquad \text{(I)}$$

This equation gives the distances r of points common to the line and the parabola from the point (α, β), hence:

(1) If the roots of this equation are equal, in other words, if the straight line is a tangent,

$$(\beta\sin\theta - 2a\cos\theta)^2 = (\beta^2 - 4a\alpha)\sin^2\theta,$$

which reduces to $\alpha - \beta\cot\theta + a\cot^2\theta = 0.$

This equation gives the directions of the tangents which can be drawn through the point (α, β) to the parabola; it is identical with the condition that the point (α, β) should lie on the tangent at the point

$$(a\cot^2\theta,\ 2a\cot\theta). \quad \text{(Cf. § 21.)}$$

(2) If the equation has equal and opposite roots, i.e. if the chords in the direction θ are bisected at the variable point (α, β), then $\beta = 2a\cot\theta.$

The properties of diameters, as before discussed, may be at once deduced from this condition.

(3) If one of the roots of this equation is infinite, i. e. if the straight line meets the curve in one point at infinity,

$$\sin^2\theta = 0.$$

This relation gives two coincident directions parallel to the axis of the parabola; hence two radii vectores can be drawn through any point to meet the parabola at infinity; these radii are equal and their directions are coincident with the direction of the axis. The geometrical meaning of this result is that all straight lines parallel to the axis are normal to the parabola at infinity; further, these are the only straight lines which meet the parabola at infinity.

Again, if the straight line meets the parabola in two coincident points at infinity,

$$\sin^2\theta = 0, \text{ and } \beta\sin\theta - 2a\cos\theta = 0,$$

therefore $\quad a = 0, \text{ or, } \beta = \infty;$

hence, no finite straight line meets a parabola at two points at infinity, i. e. a parabola has no finite asymptotes.

The straight line, $\quad 0x + 0y = a$

(which is the straight line at infinity), touches the parabola.

§ **24.** *To find the lengths of the tangents which can be drawn from any point to a parabola.*

As in the case of the general conic (II. § **16**), the lengths r of the tangents are given by the equation (I) of the last section, when the value of θ is such that the roots of the equation are equal, and therefore if t is the length of either tangent drawn from the point (x, y),

$$t^2 = \frac{y^2 - 4ax}{\sin^2\theta} = \phi\,(1 + \cot^2\theta),$$

where θ has either of the values given by the equation,

$$a\cot^2\theta - y\cot\theta + x = 0.$$

The elimination of $\cot\theta$ gives

$$a^2t^4 - \phi\,(y^2 - 2ax + 2a^2)\,t^2 + \phi^2\,(\overline{x-a}^2 + y^2) = 0,$$

which is the required equation giving the lengths of the tangents from the point (x, y) to the parabola.

If O is the point (x, y), and OP, OQ the tangents from O to the parabola, it follows that

$$a^2(OP^2+OQ^2) = \phi\,(y^2-2ax+2a^2)$$

and $$a^2\,OP^2 . OQ^2 = \phi^2\,(\overline{x-a}^2+y^2),$$

i.e. $$a^2\,OP^2 . OQ^2 = \phi^2\,OS^2$$

or, $$a\,OP . OQ = \phi\,OS.$$

NOTE. As in the general case, the coefficients of t^2 and t^0 vanish when $x = a$, $y = 0$, i.e. the tangents from the focus to the curve are zero in length.

§ **25.** *To find the equation of the normal at any point to a parabola.*

Since the equation of the tangent at the point θ is

$$\frac{x-a\cot^2\theta}{\cos\theta} = \frac{y-2a\cot\theta}{\sin\theta},$$

the equation of the normal is

$$\frac{x-a\cot^2\theta}{\sin\theta} = \frac{y-2a\cot\theta}{-\cos\theta},$$

which equation reduces to

$$a\cot^3\theta+(2a-x)\cot\theta-y = 0.$$

The equation of the normal at the point λ is, since

$$\lambda = \cot\theta,$$

$$a\lambda^3+(2a-x)\lambda-y = 0,$$

or, $$y+\lambda x = a\lambda\,(\lambda^2+2).$$

This equation is the condition that the normal at the point λ should pass through the point (x, y); if then (x, y) is regarded as a fixed point the above equation gives the values of the parameters of those points on the parabola, the normals at which to the parabola pass through the point (x, y).

NOTE (1). The equation is cubic in the variable λ, hence *three* normals can be drawn from any point to the parabola.

It has been shown above that all straight lines parallel to the axis of the parabola are normal to the curve at infinity, hence one normal through the point (x, y), corresponding to an infinite value of λ, does not appear in the equation; the complete equation is

$$0\,\lambda^4 + a\,\lambda^3 + \lambda\,(2a - x) - y = 0.$$

NOTE (2). Every cubic equation has one real root, hence one real normal can be drawn through any point to a parabola.

The other two roots are real, coincident, or imaginary, according as $27ay^2$ is greater than, equal to, or less than $4\,(x-2\,a)^3$. This condition is equivalent to the geometrical condition that the point (x, y) should lie inside, on, or outside the curve whose equation is

$$27\,ay^2 = 4\,(x - 2\,a)^3. \quad \ldots \quad \ldots \quad \text{(I)}$$

NOTE (3). If the roots of the equation

$$a\lambda^3 + (2a - x)\,\lambda - y = 0$$

are λ_1, λ_2, λ_3, since the coefficient of λ^2 is zero,

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

This is a condition, independent of the coordinates x and y, which must be satisfied by the parameters of any three points on the parabola the normals at which meet in a point. Since any two lines meet in a point, only one condition is necessary for any third line to pass through their intersection; consequently, this is the necessary and sufficient condition that the normals at the points λ_1, λ_2, λ_3 should meet in a point.

The values of the coordinates of the point of intersection of the normals at the points λ_1, λ_2, λ_3 are given by

$$x = 2a - a\,\Sigma\,\lambda_1\lambda_2 \quad\text{and}\quad y = a\,\lambda_1\lambda_2\lambda_3,$$

and provided the parameters of the feet of the normals

satisfy the condition that their sum should be zero, the point (x, y) can be found, which lies on each of the corresponding normals; it is therefore also algebraically clear that this condition is sufficient.

It follows at once that if the sum of the ordinates of three points on a parabola is zero, the normals at these points are concurrent.

NOTE (4). If two of the roots of this equation in λ are equal, two of the normals which can be drawn from the point (x, y) to the parabola coalesce.

Let $$\lambda_2 = \lambda_3 = \lambda,$$
then $$2\lambda + \lambda_1 = 0;$$
therefore the coordinates x and y are given by
$$x = 2a + 3a\lambda^2 \text{ and } y = -2a\lambda^3.$$

The elimination of λ from these two equations gives
$$27ay^2 = 4(x-2a)^3. \qquad [\textit{Vide } \text{I.}]$$
This equation, consequently, represents the locus of the intersections of coincident normals. It is called the evolute of the parabola.

NOTE (5). Three normals from a point to a parabola can only coincide when λ is zero, i.e. when the foot of each normal is the vertex of the parabola. Their point of intersection is $(2a, 0)$.

NOTE (6). The normal to the parabola at the point λ meets the curve again in a point μ given by substituting $x = a\mu^2$, $y = 2a\mu$ in the equation of the normal. This substitution gives
$$(\lambda-\mu)(\lambda^2+\mu\lambda+2) = 0,$$
i.e. $$\mu = -\frac{\lambda^2+2}{\lambda}.$$

Hence the normal at the point λ meets the parabola again at the point $-\dfrac{\lambda^2+2}{\lambda}$.

Conversely, if three normals to a parabola meet at any point (μ) on the curve, the parameters of the feet of these normals are given by

$$\lambda = \mu \text{ and } \lambda^2+\lambda\mu+2 = 0,$$

i.e.
$$\lambda = \mu, \text{ or, } -\frac{\mu \pm \sqrt{\mu^2-8}}{2}.$$

Hence, if the abscissa $a\mu^2$ of the point is greater than $8a$, three real normals can be drawn; if, however, the abscissa is less than $8a$, the two normals, other than the normal at the point itself, are imaginary; if the abscissa is equal to $8a$, these two normals coalesce, i.e. the point is then on the evolute. It follows that the parabola meets the evolute in points given by $\mu^2 = 8$, i.e. at the points $(8a, \pm 4\sqrt{2}a)$.

Again, the condition that the normals at the points λ_1, λ_2 should meet on the parabola is

$$\lambda_1 \lambda_2 = 2.$$

Hence the normals at the extremities of all chords whose equations are of the form

$$x+ky+2a = 0,$$

where k is variable, meet on the parabola. All such chords pass through the point $(-2a, 0)$.

Note (7). The normal at λ is parallel to the tangent

$$x+\frac{y}{\lambda}+\frac{a}{\lambda^2} = 0,$$

hence the ordinate of the middle point of the normal chord is

$$-\frac{2a}{\lambda}. \quad [\S\ 20, \textit{Note } 2.]$$

Example (i). *Find the orthocentre of the triangle formed by joining the feet of the normals which can be drawn from the point (X, Y) to the parabola $y^2 = 4ax$.*

Let the parameters of the feet of the normals be λ_1, λ_2, λ_3, these are the roots of the equation

$$a\lambda^3+(2a-X)\lambda-Y = 0. \quad \ldots \quad \ldots \quad \text{(I)}$$

The equation of the line through the point λ_3, perpendicular to the chord joining the points λ_1, λ_2, is

$$(\lambda_1+\lambda_2)(x-a\lambda_3^2)+2(y-2a\lambda_3)=0.$$

But $\lambda_1+\lambda_2+\lambda_3=0$, therefore this equation becomes

$$a\lambda_3^3-(x+4a)\lambda_3+2y=0.$$

If (x, y) is the orthocentre, λ_1, λ_2, and λ_3 each satisfy this equation; hence

$$a\lambda^3-(x+4a)\lambda+2y=0$$

is identical with equation (I),

$$\therefore \quad x=X-6a, \text{ and } y=-\frac{Y}{2}.$$

Example (ii). *Find the locus of the intersections of the normals at the ends of a chord passing through the fixed point (X, Y).*

Let the parameters of the ends of one chord of the system be λ_1, λ_2, the equation of the chord is

$$y(\lambda_1+\lambda_2)-2x=2a\lambda_1\lambda_2;$$

therefore, since this chord passes through the point (X, Y),

$$Y(\lambda_1+\lambda_2)-2X=2a\lambda_1\lambda_2. \quad \ldots \quad \ldots \quad \text{(I)}$$

Let the point of intersection of the normals be (x, y), and let λ be the parameter of the foot of the third normal meeting at the point (x, y).

Hence, $\qquad \lambda_1+\lambda_2=-\lambda$ and $\lambda_1\lambda_2=\dfrac{y}{a\lambda}$.

It follows that $\qquad \lambda^2Y+2X\lambda+2y=0. \quad \ldots \quad \ldots \quad$ (II)

Since (x, y) is on the normal at the point λ,

$$a\lambda^3+(2a-x)\lambda-y=0.$$

Adding twice this equation to (II)

$$2a\lambda^2+Y\lambda+2(X+2a-x)=0. \quad \ldots \quad \ldots \quad \text{(III)}$$

Eliminating λ (by cross-multiplication) from equations (II) and (III), the equation of the required locus is

$$(Y^2-4aX)\left\{Yy-2X(X-x+2a)\right\}+2\left\{Y(X-x+2a)-2ay\right\}^2=0.$$

Example (iii). *The three normals from a point P to the parabola $y^2=4ax$, and the line through P parallel to the axis of the parabola, form an harmonic pencil. Find the locus of P.*

Let the parameters of the feet of the normals be λ_1, λ_2, λ_3.

By hypothesis the lines $y+\lambda_1x=0$, $y+\lambda_2x=0$, $y+\lambda_3x=0$, and $y=0$ form an harmonic pencil.

Hence $$2\lambda_1\lambda_2 = \lambda_3(\lambda_1+\lambda_2).$$

Therefore, since $\lambda_1+\lambda_2+\lambda_3 = 0,\ 2\lambda_1\lambda_2 = -\lambda_3^2.$

The coordinates of the point P are given by

$$\frac{x-2a}{a} = -\Sigma\lambda_1\lambda_2 = -\lambda_1\lambda_2+\lambda_3^2 = -3\lambda_1\lambda_2,$$

$$\frac{y}{a} = \lambda_1\lambda_2\lambda_3.$$

Therefore $$\frac{y^2}{a^2} = \lambda_1^2\lambda_2^2\lambda_3^2 = -2(\lambda_1\lambda_2)^3 = 2\left(\frac{x-2a}{3a}\right)^3.$$

The required locus is therefore

$$27ay^2 = 2(x-2a)^3.$$

§ **26.** *The intersections of a circle and the parabola.*

The equation of any circle can be put in the form

$$x^2+y^2+2gx+2fy+c = 0. \quad \dots \quad \dots \quad \text{(I)}$$

This circle meets the parabola in points whose parameters are given by the equation obtained by substituting $x = a\lambda^2$, $y = 2a\lambda$ in equation (I) of the circle; this equation is

$$a^2\lambda^4+(4a^2+2ga)\lambda^2+4fa\lambda+c = 0. \quad \dots \quad \text{(II)}$$

NOTE (1). This equation is quartic in the variable λ; hence every circle meets the parabola in four points, which may be all real, or two real and two imaginary, or all imaginary.

NOTE (2). If the values of λ given by this equation are $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$, since the coefficient of λ^3 in the equation is zero, $$\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 0.$$

Hence, if four points on a parabola are concyclic, the sum of the parameters of the points is zero, or, the sum of the ordinates of the four points is zero. This condition is also sufficient, for provided the sum of the parameters is zero the coefficients in the equation of the circle are given by three independent relations, viz.

$$2g = a\Sigma\lambda_1\lambda_2-4a,$$

$$4f = -a\Sigma\lambda_1\lambda_2\lambda_3,$$

$$c = a^2\lambda_1\lambda_2\lambda_3\lambda_4.$$

These three relations give the centre and radius of the circle which can be drawn through four points, the sum of the parameters of which is zero.

The directions of the chords joining the points λ_1, λ_2 and λ_3, λ_4 respectively, are $\frac{1}{2}(\lambda_1+\lambda_2)$ and $\frac{1}{2}(\lambda_3+\lambda_4)$; but if these four points are concyclic

$$\lambda_1+\lambda_2 = -(\lambda_3+\lambda_4);$$

therefore these chords are equally inclined to the axis of the parabola. Hence the common chords of any circle and a parabola are equally inclined in pairs to the axis.

NOTE (3). If two roots of equation (II) of this section are equal, two of the intersections of the circle and parabola coincide; in other words, the circle touches the parabola, one of their common chords becoming a common tangent to the circle and parabola at their point of contact. This tangent and the common chord of the circle and parabola are equally inclined to the axis.

NOTE (4). If three roots of this equation are equal, three points of intersection of the circle and parabola coincide, and the circle is said to osculate the parabola at this point.

If λ is the parameter of the three points which coalesce, and λ_1 the fourth point of intersection of the circle and the parabola, then
$$3\lambda+\lambda_1 = 0.$$

Therefore the osculating circle at the point λ meets the parabola again at the point -3λ. The equation of the chord of curvature is consequently

$$\lambda y+x = 3a\lambda^2.$$

The centre and radius of curvature at any point λ on the parabola are given by

$$-g = 3a\lambda^2+2a,$$
$$-f = -2a\lambda^3, \quad c = -3a^2\lambda^4,$$
$$\rho^2 = f^2+g^2-c = 4a^2(1+\lambda^2)^3;$$

i.e.
$$\rho = 2a(1+\lambda^2)^{\frac{3}{2}}.$$

Two simple deductions may be made from these results:

(1) The equation of the chord of curvature is quadratic in the variable λ, hence through any point (x, y) two chords of curvature pass.

(2) If the length of the radius of curvature is given, the points on the parabola at which the radius of curvature has this value are given by the equation

$$4a^2(\lambda^2+1)^3 = \rho^2.$$

This is a cubic equation in λ^2, two of its roots are imaginary and one real; corresponding to the real value there are two equal and opposite values of λ. Hence there are two circles of curvature of given radius to every parabola, and these circles are symmetrically placed with regard to the axis of the parabola.

The equation of the circle of curvature at the point λ can now be obtained by substituting the values of f, g, and c found above; this equation is

$$x^2+y^2-2(3a\lambda^2+2a)x+4a\lambda^3y-3a^2\lambda^4 = 0.$$

This equation is quartic in the variable λ, and consequently four circles of curvature meet at any point (x, y); the parameters of the points at which these four circles osculate the parabola are given by this equation. If these are λ_1, λ_2, λ_3, λ_4, then, since the coefficient of λ is zero,

$$\Sigma\lambda_1\lambda_2\lambda_3 = 0, \text{ or, } \Sigma\frac{1}{\lambda} = 0.$$

This is a necessary, but not a sufficient, condition that the circles of curvature at four points on the parabola should all pass through the same point. The coefficients of the equation contain only two independent quantities, x and y, and there are three relations between these and the symmetrical functions of the roots, viz.

$$x = \frac{a}{2}\Sigma\lambda_1\lambda_2;\ y = \frac{3a}{4}\Sigma\lambda;\ \text{and } x^2+y^2-4ax = -3a^2\lambda_1\lambda_2\lambda_3\lambda_4;$$

if x and y are eliminated from these three equations, a second condition independent of x and y is obtained. If, further, λ_4 is eliminated from this condition and the condition found above, the resulting equation will be the necessary and sufficient condition that the circles of curvature at the points λ_1, λ_2, λ_3 should meet in one point.

NOTE (5). If there are two pairs of equal roots, the circle touches the parabola at two points, i.e. has double contact with it.

Let $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_4$.

Now $\Sigma\lambda = 0$, therefore $\lambda_1 = -\lambda_2$.

Hence, if a circle has double contact with a parabola, the chord of contact is perpendicular to the axis.

Also, since $f = -\frac{a}{4}\Sigma\lambda_1\lambda_2\lambda_3 = 0$, the centre of the circle is on the axis, which is also clear from symmetry.

Further, the corresponding value of g gives

$$\lambda_1^2 = -2 - \frac{g}{a}.$$

If, then, λ_1 is real the abscissa of the centre $(-g)$ must be positive and greater than $2a$.

NOTE (6). The four roots of this equation can only be equal when λ is zero, therefore only one circle can meet the parabola in four coincident points. This circle has four-point contact with the parabola at the vertex, and its centre is the point $(2a, 0)$, it is the limiting case of the circles discussed in Note (5).

Example (i). *A circle is described on a chord of a parabola whose equation is $Ax + By + a = 0$ as diameter; find the equation of the other common chord of the circle and the parabola.*

Let λ_1, λ_2 be the parameters of the extremities of the given chord, λ_1, λ_2 are given by the equation

$$A\lambda^2 + 2B\lambda + 1 = 0.$$

If λ is either of the other points of intersection of the parabola

and the circle on the chord joining the points λ_1, λ_2 as diameter, since the joins of the pairs of points $\lambda\lambda_1$, $\lambda\lambda_2$ are at right angles (for the angle in a semicircle is a right angle),

$$(\lambda+\lambda_1)(\lambda+\lambda_2)+4=0.$$

Hence the parameters λ_3, λ_4 of the other points of intersection are given by the equation

$$\lambda^2+\lambda(\lambda_1+\lambda_2)+\lambda_1\lambda_2+4=0.$$

Hence $$\lambda_3+\lambda_4=-(\lambda_1+\lambda_2)=\frac{2B}{A},$$

$$\lambda_3\lambda_4=\lambda_1\lambda_2+4=\frac{1}{A}+4.$$

Therefore the equation of the chord joining the points λ_3, λ_4 is

$$By-Ax=(1+4A)a.$$

Example (ii). *A triangle is inscribed in a parabola with its orthocentre at the focus. Prove that its circumscribing circle touches the tangent at the vertex.*

Let the vertices of the triangle be the points λ_1, λ_2, λ_3. By hypothesis the focal chord through the point λ_3 is perpendicular to the chord joining the points λ_1, λ_2, therefore since the other extremity of the focal chord is $-\dfrac{1}{\lambda_3}$,

$$\left(\lambda_1+\lambda_2\right)\left(\lambda_3-\frac{1}{\lambda_3}\right)+4=0, \qquad \ldots \qquad \ldots \qquad \text{(I)}$$

and, symmetrically, $$\left(\lambda_2+\lambda_3\right)\left(\lambda_1-\frac{1}{\lambda_1}\right)+4=0. \qquad \ldots \qquad \ldots \qquad \text{(II)}$$

(1) Subtract (II) from (I), and divide the result by $\lambda_1-\lambda_3$, then

$$\lambda_1\lambda_2\lambda_3+\lambda_1+\lambda_2+\lambda_3=0.$$

(2) Subtract λ_3 times (I) from λ_1 times (II), then

$$\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1=-5.$$

Let the equation of the circumscribing circle be

$$x^2+y^2+2gx+2fy+c=0.$$

If the fourth point of intersection is λ_4, then λ_1, λ_2, λ_3, λ_4 are roots of the equation

$$a^2\lambda^4+(4a^2+2ag)\lambda^2+4af\lambda+c=0;$$

therefore $$\lambda_4=-(\lambda_1+\lambda_2+\lambda_3),$$

and $$c=a^2\lambda_1\lambda_2\lambda_3\lambda_4=a^2\lambda_4^2$$

from the first condition found above.

Further, $$-\frac{4f}{a} = \lambda_4(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)+\lambda_1\lambda_2\lambda_3,$$
$$= -5\lambda_4+\lambda_4 = -4\lambda_4,$$
using the second condition found above.

It follows at once that $$c = f^2,$$
which is the condition that the circle should touch the tangent at the vertex $(x = 0)$.

§ **27.** *Relations between the points of intersection of the tangents and normals at any two points on the parabola.*

Let (x, y) be the point of intersection of the tangents at two points and (ξ, η) that of the normals.

The points of contact of the tangents from the point (x, y) to the parabola are given by
$$a\lambda^2-\lambda y+x = 0, \quad \dots \quad \dots \quad \text{(I)}$$
and the feet of the normals meeting at (ξ, η) are given by
$$a\lambda^3+(2a-\xi)\lambda-\eta = 0. \quad \dots \quad \dots \quad \text{(II)}$$

Hence two of the roots of equation (II) are the same as the roots of equation (I). Subtract λ times equation (I) from equation (II), it follows that the two equations
$$\lambda^2 y+\lambda(2a-\xi-x)-\eta = 0,$$
$$\lambda^2 a-\lambda y+x = 0$$
have the same roots.

Hence $$\frac{y}{a} = \frac{\xi+x-2a}{y} = -\frac{\eta}{x}. \quad \dots \quad \dots \quad \text{(III)}$$

Supposing that the intersections of the tangents lie on some locus $$f(x, y) = 0,$$
the corresponding locus of the intersections of the normals is obtained by eliminating x and y from the equation $f(x, y) = 0$ and equations (III).

Conversely, if the locus of the intersections of the normals is given, the locus of the intersections of the corresponding tangents is obtained by eliminating ξ and η.

The result in the second case is in general easily obtained, for

$$\xi = \frac{y^2}{a} - x + 2a \quad \text{and} \quad \eta = -\frac{xy}{a},$$

therefore if the locus of the intersections of the normals at two points is

$$f(\xi, \eta) = 0,$$

the equation of the locus of the intersections of corresponding tangents is

$$f\left(\frac{y^2}{a} - x + 2a, \; -\frac{xy}{a}\right) = 0.$$

Example (i). *Normals are drawn to a parabola from points on the lines $x = h$, $y = k$; find in each case the corresponding locus of the intersections of tangents.*

These loci are clearly $y^2 - ax + 2a^2 = ah$,

and $$xy = -ak.$$

Example (ii). *Tangents are drawn to a parabola from points on a line parallel to the axis; prove that the normals at their points of contact intersect on a fixed straight line.*

Let $y = c$ be the given line and $y^2 = 4ax$ the parabola.

The required locus is obtained by eliminating x and y from the equations

$$y = c; \quad \frac{y}{a} = \frac{\xi + x - 2a}{y} = -\frac{\eta}{x}.$$

Hence $$x = \frac{c^2}{a} + 2a - \xi \text{ and } x = -\frac{a\eta}{c},$$

therefore the required locus is

$$\frac{c^2}{a} + 2a - \xi = -\frac{a\eta}{c},$$

or, $$x - \frac{ay}{c} = \frac{c^2}{a} + 2a.$$

Illustrative Examples.

Example (i). *A circle cuts a parabola in four points; if the normals at three of these points are concurrent, prove that the circle passes through the vertex, and find its equation.*

Let the four points be $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

Then, because they are concyclic,

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 0.$$

But the normals at three of these points are concurrent, hence

$$\lambda_1+\lambda_2+\lambda_3 = 0.$$

Consequently $\lambda_4 = 0$, i. e. the circle passes through the vertex.

If the normals meet at the point (ξ, η), the parameters $\lambda_1, \lambda_2, \lambda_3$ are given by

$$a\lambda^3+(2a-\xi)\lambda-\eta = 0. \quad \ldots \quad \ldots \quad \text{(I)}$$

The intersections of the circle

$$x^2+y^2+2gx+2fy = 0$$

and the parabola are given by

$$a\lambda^3+\lambda(4a+2g)+4f = 0. \quad \ldots \quad \ldots \quad \text{(II)}$$

If this is the required circle, the roots of this equation are $\lambda_1, \lambda_2, \lambda_3$.

Hence, comparing equations (I) and (II),

$$2g = -(\xi+2a),$$

$$2f = -\frac{\eta}{2}.$$

The circle through the feet of the normals which meet at the point (ξ, η) is accordingly

$$x^2+y^2-(\xi+2a)x-\frac{\eta}{2}y = 0.$$

Example (ii). *If OP, OQ are the tangents from any point O to the parabola $y^2-4ax=0$, and OL, OM, ON the normals meeting at O, prove that* $\quad OL.OM.ON = a\,OP.OQ.$

It has been shown (§ 24), that

$$a\,OP.OQ = (y^2-4ax)\,OS,$$

where (x, y) is the point O.

The feet of the normals are given by the equation

$$a\lambda^3+(2a-x)\lambda-y = 0$$
$$\equiv a(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3),$$

where $\lambda_1, \lambda_2, \lambda_3$ are roots of the equation.

Hence $\quad \Sigma\lambda = 0;\ \therefore\ \Sigma\lambda^2 = -2\Sigma\lambda_1\lambda_2 = \dfrac{2x-4a}{a};$

$$(\Sigma\lambda_1\lambda_2)^2 = \Sigma\lambda_1^2\lambda_2^2 = \left(\frac{x-2a}{a}\right)^2.$$

$$OL = \sqrt{\{(x-a\lambda_1^2)^2+(y-2a\lambda_1)^2\}}$$
$$= \frac{y-2a\lambda_1}{\lambda_1}\sqrt{1+\lambda_1^2} = \frac{2a}{\lambda_1}\left(\frac{y}{2a}-\lambda_1\right)\sqrt{1+\lambda_1^2};$$

therefore

$$OL.OM.ON = \frac{8a^3}{\lambda_1\lambda_2\lambda_3}\left(\frac{y}{2a}-\lambda_1\right)\left(\frac{y}{2a}-\lambda_2\right)\left(\frac{y}{2a}-\lambda_3\right)$$
$$\{1+\Sigma\lambda^2+\Sigma\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_2^2\lambda_3^2\}^{\frac{1}{2}},$$
$$=\frac{8a^3}{y}\left\{\frac{y^3}{8a^3}+(2a-x)\frac{y}{2a^2}-\frac{y}{a}\right\}\left\{x^2+y^2-2ax+a^2\right\}^{\frac{1}{2}},$$
$$=(y^2-4ax)\,OS.$$

Example (iii). *From a point P three normals are drawn to a parabola in such a way that the rectangular hyperbola, passing through P and the feet of the three normals, passes also through the vertex. Find the locus of P.*

If P is the point (X, Y), the feet of the normals are given by

$$a\lambda^3+(2a-X)\lambda-Y=0. \quad \dots \quad \dots \quad \text{(I)}$$

Any rectangular hyperbola through the vertex is

$$x^2-y^2+2hxy+2gx+2fy=0.$$

This meets the parabola at points given by the equation

$$a\lambda^3+4ha\lambda^2+(2g-4a)\lambda+2f=0. \quad \dots \quad \dots \quad \text{(II)}$$

If these points are also the feet of the normals given by (I), it follows that $h=0,\ 2g-4a=2a-X$, and $2f=-Y$.

The equation of the rectangular hyperbola is then

$$x^2-y^2+(6a-X)x-Yy=0.$$

But by hypothesis (X, Y) lies on this hyperbola, therefore

$$X^2-Y^2+(6a-X)X-Y^2=0,$$

i.e. $$Y^2=3aX.$$

Hence the locus of P is the parabola

$$y^2=3ax.$$

Example (iv). *Two tangents TP, TP' to a parabola meet the tangent at the vertex in Q, Q'. Prove that the radius of the circle TQQ' is $\frac{1}{8}f_1^{\frac{1}{2}}f_2^{\frac{1}{2}}$, where f_1, f_2 are the focal chords parallel to TP and TP'.*

Let P be the point λ, P' the point μ.

Then $$f_1=4SP=4a(1+\lambda^2),$$

and similarly $$f_2=4a(1+\mu^2).$$

The equation of the tangent at the point λ is

$$\lambda y - x = a\lambda^2,$$

hence Q is the point $(0, a\lambda)$, and Q' the point $(0, a\mu)$.

If θ is the angle between the tangents at the points λ, μ

$$\tan\theta = \frac{\lambda-\mu}{1+\lambda\mu};$$

$$\therefore \quad \sin\theta = \frac{\lambda-\mu}{\sqrt{(1+\lambda^2)(1+\mu^2)}} = \frac{QQ'}{a\sqrt{(1+\lambda^2)(1+\mu^2)}}.$$

But the radius of the circle TQQ'

$$= \frac{QQ'}{2\sin\theta} = \frac{a\sqrt{(1+\lambda^2)(1+\mu^2)}}{2} = \frac{f_1^{\frac{1}{2}}f_2^{\frac{1}{2}}}{8}.$$

CHAPTER IV

THE ELLIPSE AND HYPERBOLA REFERRED TO THEIR AXES

§ **28.** The equation of an ellipse, whose major and minor axes are $2a$ and $2b$ respectively, referred to its axes as axes of coordinates, is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \quad \ldots \quad \ldots \quad \text{(I)}$$

Any point whose coordinates are of the form $(a\cos\theta, b\sin\theta)$ lies on the ellipse, for if the values of these coordinates is substituted in the equation of the ellipse, the result is identically true for all values of θ.

Further, it is evident from equation (I) that for real points on the ellipse the numerical values of x and y cannot exceed a and b respectively, hence all possible values of x and y which satisfy equation (I) of the curve can be represented by $a\cos\theta$ and $b\sin\theta$ for some value of θ. Consequently any point on the ellipse can be represented by $(a\cos\theta, b\sin\theta)$; this point will be referred to as 'the point θ.'

The equation of the circle described on the major axis of the ellipse as diameter is

$$x^2+y^2=a^2,$$

and any point on this circle can be represented by

$$(a\cos\theta, a\sin\theta).$$

Hence to each point θ on the ellipse there corresponds a point θ on this circle, and the abscissae of these two points are the same.

Such points are called corresponding points, and the angle θ, which is the inclination of the radius of the circle to the axis of x, is called the eccentric angle of the point. The circle is called the auxiliary circle.

The equations obtained by using the eccentric angle are more complicated than those obtained by use of the single variable in the case of the parabola; the reason is that $\cos\theta$ and $\sin\theta$ can neither be expressed rationally in terms of the other. It is often advantageous to use the corresponding exponential forms, viz. :—

$$x = \tfrac{1}{2}a\,(e^{\iota\theta}+e^{-\iota\theta}), \quad y = \frac{b}{2\iota}(e^{\iota\theta}-e^{-\iota\theta}), \quad [\iota = \sqrt{-1}].$$

These coordinates involve the use of imaginary quantities, which is objected to by many; their application will, however, be illustrated later in this chapter.

§ **29.** *To find the intersections of a straight line and the ellipse.*

The equation of any straight line may be written in the form

$$A\frac{x}{a}+B\frac{y}{b}+C=0.$$

The points in which this straight line meets the ellipse are given by the equation obtained by substituting $x = a\cos\theta$ and $y = b\sin\theta$ in the equation of the line; this gives

$$A\cos\theta+B\sin\theta+C=0.$$

This equation is quadratic in any trigonometrical function of θ; hence, as in the general case, the straight line meets the ellipse in two points θ_1, θ_2, the values θ_1 and θ_2 being given by this equation.

The values of A, B, and C in terms of θ_1, θ_2 are most easily obtained by cross-multiplication from the equations

$$A\cos\theta_1+B\sin\theta_1+C=0,$$
$$A\cos\theta_2+B\sin\theta_2+C=0$$

which are the conditions that the points θ_1, θ_2 should lie on the given straight line. Hence

$$\frac{A}{\cos\frac{1}{2}(\theta_1+\theta_2)}=\frac{B}{\sin\frac{1}{2}(\theta_1+\theta_2)}=\frac{-C}{\cos\frac{1}{2}(\theta_1-\theta_2)}.$$

Consequently the equation of the chord of the ellipse joining the points θ_1, θ_2 is

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta_1+\theta_2)+\frac{y}{b}\sin\tfrac{1}{2}(\theta_1+\theta_2)=\cos\tfrac{1}{2}(\theta_1-\theta_2).$$

NOTE (1). The equation of the tangent at the point θ is obtained by putting $\theta_1=\theta_2=\theta$ in the equation of the chord, which gives

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1.$$

NOTE (2). If the chord passes through the centre of the ellipse, i.e. through the origin,

$$\cos\frac{\theta_1-\theta_2}{2}=0,\quad\text{or,}\quad\theta_1-\theta_2=\pi;$$

hence θ and $\pi+\theta$ represent the extremities of a diameter.

NOTE (3). The eccentric angles of the ends of any chord parallel to the fixed straight line

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=0$$

satisfy the relation

$$\tan\frac{\theta_1+\theta_2}{2}=\tan\alpha,$$

or,

$$\theta_1+\theta_2=2\alpha.$$

The coordinates of the middle points of such chords are given by

$$\frac{x}{a}=\tfrac{1}{2}(\cos\theta_1+\cos\theta_2)=\cos\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_1-\theta_2}{2},$$

$$\frac{y}{b}=\tfrac{1}{2}(\sin\theta_1+\sin\theta_2)=\sin\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_1-\theta_2}{2}.$$

This point lies on the line whose equation is

$$\frac{y}{b}=\frac{x}{a}\tan\frac{\theta_1+\theta_2}{2},$$

or,

$$\frac{y}{b}=\frac{x}{a}\tan\alpha.$$

Hence all chords of the ellipse drawn parallel to the diameter

$$\frac{y}{b}+\frac{x}{a}\cot\alpha=0 \qquad \ldots \qquad \ldots \qquad \text{(I)}$$

are bisected by the diameter

$$\frac{y}{b}-\frac{x}{a}\tan\alpha=0. \qquad \ldots \qquad \ldots \qquad \text{(II)}$$

It is evident, similarly, that chords parallel to the second diameter are bisected by the first. These diameters are called conjugate diameters.

The pair of conjugate diameters, whose equations are (II) and (I), meet the ellipse in points P, P' and D, D' respectively, whose coordinates are

$$(\pm a\cos\alpha,\ \pm b\sin\alpha)\ (\mp a\sin\alpha,\ \pm b\cos\alpha).$$

Hence $\qquad CP^2=a^2\cos^2\alpha+b^2\sin^2\alpha,$

$$CD^2=a^2\sin^2\alpha+b^2\cos^2\alpha,$$

therefore $\qquad CP^2+CD^2=a^2+b^2,$

or, the sum of the squares on any two conjugate diameters is constant.

It is further evident, from the form of the coordinates, that if P is the point α, D is the point $\alpha\pm\frac{\pi}{2}$.

NOTE (4). The straight lines

$$Ax+By+C=0,$$
$$A'x+B'y+C'=0$$

are parallel to conjugate diameters, if

$$a^2AA'+b^2BB'=0.$$

It is useful to note that the middle point of the chord of the ellipse, whose equation is

$$Ax+By+C=0,$$

is given by the intersection of this straight line and the diameter

$$\frac{Bx}{a^2}-\frac{Ay}{b^2}=0.$$

NOTE (5). The equation of the tangent to an ellipse at the point $P(\theta)$ can now be written

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{r}{CD},$$

where CD is the semi-diameter conjugate to CP.

The equation of the normal to the ellipse at the point $P(\theta)$ is, accordingly,

$$\frac{x-a\cos\theta}{b\cos\theta}=\frac{y-b\sin\theta}{a\sin\theta}=\frac{r}{CD},$$

r being positive when measured outwards; this equation may also be written

$$ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2.$$

Example (i). *To find the locus of the middle points of normal chords of an ellipse.*

The equation of the normal at the point θ is

$$ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2.$$

The middle point of the normal chord is the intersection of this line and the diameter

$$b\operatorname{cosec}\theta\frac{x}{a^2}+a\sec\theta\frac{y}{b^2}=0.$$

Hence $$\tan\theta=-\frac{b^3x}{a^3y}.$$

Eliminating θ, the required locus is

$$(b^2x^2+a^2y^2)^2(b^6x^2+a^6y^2)=a^4b^4(a^2-b^2)^2x^2y^2.$$

Example (ii). *If the normal at P to an ellipse meets the axes in the points G, G' and O is a point on it such that*

$$\frac{2}{PO}=\frac{1}{PG}+\frac{1}{PG'},$$

then will any chord through O subtend a right angle at P.

Let P be the point α, the equation to the normal at P is

$$\frac{x-a\cos\alpha}{b\cos\alpha}=\frac{y-b\sin\alpha}{a\sin\alpha}=\frac{r}{CD}.$$

Putting $y=0$ and $x=0$ successively in these equations, the values of r are PG and PG' respectively, hence

$$PG=-\frac{b}{a}CD \text{ and } PG'=-\frac{a}{b}CD.$$

By hypothesis it follows that

$$PO = -\frac{2ab}{a^2+b^2}CD.$$

Since the point O is on the normal at P, its coordinates are given by

$$\frac{x-a\cos\alpha}{b\cos\alpha} = \frac{y-b\sin\alpha}{a\sin\alpha} = -\frac{2ab}{a^2+b^2};$$

$$\therefore \quad x = \frac{a^2-b^2}{a^2+b^2}a\cos\alpha, \quad y = \frac{b^2-a^2}{a^2+b^2}b\sin\alpha.$$

Let the extremities of any chord through the point O be θ_1, θ_2, the equation of the chord is

$$\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2} + \frac{y}{b}\sin\frac{\theta_1+\theta_2}{2} = \cos\frac{\theta_1-\theta_2}{2}.$$

Since the point O is on this line,

$$\cos\alpha\cos\frac{\theta_1+\theta_2}{2} - \sin\alpha\sin\frac{\theta_1+\theta_2}{2} = \frac{a^2+b^2}{a^2-b^2}\cos\frac{\theta_1-\theta_2}{2}.$$

This condition may be written

$$\frac{\cos\left(\frac{\theta_1+\alpha}{2}+\frac{\theta_2+\alpha}{2}\right)}{\cos\left(\frac{\theta_1+\alpha}{2}-\frac{\theta_2+\alpha}{2}\right)} = \frac{a^2+b^2}{a^2-b^2},$$

i.e. $$\tan\frac{\theta_1+\alpha}{2}\tan\frac{\theta_2+\alpha}{2} = -\frac{b^2}{a^2},$$

which is the condition that the chords joining the pairs of points θ_1, α and θ_2, α should be perpendicular.

§ **30.** The equation of the tangent to the ellipse at the point θ has been shown to be

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$

If the point (x, y) is considered fixed, this equation expresses the condition that the tangent at a variable point θ on the ellipse should pass through the point (x, y). Hence this equation gives the eccentric angles of the points of contact of tangents from the point (x, y) to the ellipse.

This equation can also be written

$$\left(1+\frac{x}{a}\right)\tan^2\frac{\theta}{2} - \frac{2y}{b}\tan\frac{\theta}{2} + 1 - \frac{x}{a} = 0,$$

which is quadratic; this means that two tangents can be drawn from any point to an ellipse.

It may be deduced, as in the case of the general conic (§ 15), that since tangents from points outside, on, or inside an ellipse respectively are real, coincident, or imaginary, the point (x, y) lies outside, on, or inside the ellipse according as $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ is positive, zero, or negative.

If θ_1, θ_2 be the eccentric angles of the points of contact of tangents from the point (x, y) to the ellipse, since they satisfy the equation

$$\left(1+\frac{x}{a}\right)\tan^2\frac{\theta}{2}-\frac{2y}{b}\tan\frac{\theta}{2}+1-\frac{x}{a}=0,$$

it follows that

$$\tan\frac{\theta_1}{2}+\tan\frac{\theta_2}{2}=\frac{\frac{2y}{b}}{1+\frac{x}{a}}, \quad \text{and} \quad \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}=\frac{1-\frac{x}{a}}{1+\frac{x}{a}}.$$

The point of intersection (x, y) of tangents at the points θ_1, θ_2 is accordingly

$$\left\{a\,\frac{1-\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}}{1+\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}},\quad b\,\frac{\tan\frac{\theta_1}{2}+\tan\frac{\theta_2}{2}}{1+\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}}\right\}.$$

This may also be written

$$\left\{a\,\frac{\cos\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}},\quad b\,\frac{\sin\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}\right\}.$$

The equation of the chord joining the points θ_1, θ_2 is

$$\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2}+\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1-\theta_2}{2};$$

hence, if the tangents at the points θ_1, θ_2 meet at the point (X, Y), the equation of this chord, which is the *polar* of the point (X, Y), may be written

$$\frac{xX}{a^2}+\frac{yY}{b^2}=1.$$

Again, the angle γ which the tangent at the point θ makes with the axis of x is given by

$$\tan\gamma=-\frac{b}{a}\cot\theta.$$

But the equation of the tangent at the point θ,

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1,$$

may be put in the form,

$$\left(1-\frac{y^2}{b^2}\right)\tan^2\theta-\frac{2xy}{ab}\tan\theta+1-\frac{x^2}{a^2}=0.$$

Hence the inclinations to the axis of x of tangents from the point (x, y) to the ellipse are given by the equation

$$(a^2-x^2)\tan^2\gamma+2xy\tan\gamma+(b^2-y^2)=0.$$

If, then, the tangents from the point (x, y) make angles α_1, α_2 with the axis of x,

$$\tan\gamma_1+\tan\gamma_2=-\frac{2xy}{a^2-x^2},$$

and
$$\tan\gamma_1\tan\gamma_2=\frac{b^2-y^2}{a^2-x^2}.$$

It can at once be deduced that

(1) The angle between the tangents from the point (x, y) to the ellipse is

$$\tan^{-1}\frac{2ab\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^{\frac{1}{2}}}{a^2+b^2-x^2-y^2}.$$

(2) The locus of points, the tangents from which to the ellipse make an angle α with each other, is

$$(x^2+y^2-a^2-b^2)^2 \tan^2\gamma = 4a^2b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).$$

(3) In particular, the locus of the intersections of orthogonal tangents is $x^2+y^2=a^2+b^2$, which is the director circle.

§ **31.** The equation to the normal at the point θ has been shown to be

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2.$$

If the point (x, y) is regarded as fixed, this equation represents the condition that the normal at a variable point θ should pass through the point (x, y); conversely, the equation gives the eccentric angles of the feet of the normals which can be drawn from the point (x, y) to the ellipse.

This equation can be written in the three following forms, where $c^2 \equiv a^2-b^2$.

$$\text{I.}\quad c^4\cos^4\theta - 2c^2ax\cos^3\theta + (a^2x^2+b^2y^2-c^4)\cos^2\theta + 2c^2ax\cos\theta - a^2x^2 = 0.$$

$$\text{II.}\quad c^4\sin^4\theta + 2c^2by\sin^3\theta + (a^2x^2+b^2y^2-c^4)\sin^2\theta - 2c^2by\sin\theta - b^2y^2 = 0.$$

$$\text{III.}\quad by\tan^4\frac{\theta}{2} + 2(ax+c^2)\tan^3\frac{\theta}{2} + 2(ax-c^2)\tan\frac{\theta}{2} - by = 0.$$

Since each of these equations is quartic, it follows that in general four normals can be drawn from any point to an ellipse, the eccentric angles of their points of intersection with the ellipse being given by either of these equations.

Note (1). If $\theta_1, \theta_2, \theta_3, \theta_4$ are the eccentric angles of the feet of the normals which can be drawn from any point (x, y) to an ellipse, since the coefficient of $\tan^2\frac{\theta}{2}$ in equation III is zero, and also the coefficient of $\tan^4\frac{\theta}{2}$ is

equal and opposite to the absolute term, it follows that

$$(1) \quad \Sigma \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = 0,$$

and $$(2) \quad \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\tan\frac{\theta_4}{2} = -1.$$

These two conditions are independent of the coordinates x and y, and since two conditions are sufficient in order that four lines should meet in a point, they are the necessary and sufficient conditions that the normals at the four points θ_1, θ_2, θ_3, θ_4 should be concurrent.

It follows immediately that

$$(1) \quad \tan\tfrac{1}{2}(\theta_1+\theta_2+\theta_3+\theta_4) = \infty,$$

or, $$\theta_1+\theta_2+\theta_3+\theta_4 = (2n+1)\pi.$$

Also,

$$(2) \quad \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}+\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}+\tan\frac{\theta_3}{2}\tan\frac{\theta_1}{2}$$

$$= -\tan\frac{\theta_4}{2}\left(\tan\frac{\theta_1}{2}+\tan\frac{\theta_2}{2}+\tan\frac{\theta_3}{2}\right)$$

$$= \cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2}+\cot\frac{\theta_2}{2}\cot\frac{\theta_3}{2}+\cot\frac{\theta_3}{2}\cot\frac{\theta_1}{2}.$$

But $$\cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2}-\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{2(\cos\theta_1+\cos\theta_2)}{\sin\theta_1\sin\theta_2}.$$

Hence $$\Sigma \sin\theta_3(\cos\theta_1+\cos\theta_2) = 0,$$

or, $$\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1) = 0.$$

This is accordingly the necessary and sufficient condition that the normals at the three points θ_1, θ_2, θ_3 should be concurrent.

Example (i). *If from any point four normals are drawn to an ellipse, meeting one of the axes in G_1, G_2, G_3, G_4, then will*

$$\Sigma\frac{1}{CG} = \frac{4}{\Sigma CG}.$$

Put $y = 0$ in the equation of the normal, then

$$CG = \frac{a^2-b^2}{a}\cos\theta.$$

Hence it is required to prove that

$$\Sigma\cos\theta\,\Sigma\sec\theta = 4,$$

where θ has the four values for points the normals at which are concurrent.

From equation I,

$$\Sigma\cos\theta = \frac{2ax}{c^2},\quad \Sigma\sec\theta = \frac{2c^2}{ax},$$

hence the required condition.

The same property may be proved for points on the minor axis.

Example (ii). *The normals at $PQRS$ meet in a point; and $P'Q'R'S'$ are the points on the auxiliary circle corresponding to $PQRS$ respectively. If straight lines are drawn through $PQRS$ parallel to $P'C$, $Q'C$, $R'C$, $S'C$, they are concurrent.*

Let P be the point θ, then P' is $(a\cos\theta, a\sin\theta)$.

The equation of $P'C$ is

$$x\sin\theta - y\cos\theta = 0.$$

Hence the equation of a line through P parallel to this is

$$(x-a\cos\theta)\sin\theta-(y-b\sin\theta)\cos\theta = 0,$$

or,
$$x\sin\theta-y\cos\theta-(a-b)\sin\theta\cos\theta = 0.$$

Conversely this gives the points θ, in which lines of this form through the point (x, y) intersect the ellipse. Writing t for $\tan\frac{\theta}{2}$, the equation gives

$$yt^4+2t^3(x+a-b)+2t(x-a+b)-y = 0.$$

The conditions therefore that lines through the points θ_1, θ_2, θ_3, θ_4 of this form should be concurrent are,

$$\Sigma t_1 t_2 = 0 \text{ and } t_1 t_2 t_3 t_4 = -1.$$

These are identical with the conditions that the normals at these points should be concurrent, and, hence, the proposition is true.

NOTE (2). The coordinates of the point of intersection of the normals at any four points whose eccentric angles θ_1, θ_2, θ_3, θ_4 satisfy the conditions that these normals should

be concurrent, are

$$x = \frac{a^2-b^2}{2a} \Sigma \cos\theta, \quad \ldots \quad \ldots \quad \text{(I)}$$

$$y = \frac{b^2-a^2}{2b} \Sigma \sin\theta. \quad \ldots \quad \ldots \quad \text{(II)}$$

NOTE (3). Let OP, OQ, OR, OT be four normals to the ellipse from the point O, and let θ_1, θ_2, θ_3, θ_4 be the eccentric angles of the feet of these normals.

The equations of the chords PQ and RT are

$$\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2}+\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1-\theta_2}{2},$$

$$\frac{x}{a}\cos\frac{\theta_3+\theta_4}{2}+\frac{y}{b}\sin\frac{\theta_3+\theta_4}{2}=\cos\frac{\theta_3-\theta_4}{2}.$$

Now, $$2\cos\frac{\theta_1-\theta_2}{2}\cos\frac{\theta_3-\theta_4}{2}$$

$$= \cos\tfrac{1}{2}(\theta_1+\theta_3-\theta_2-\theta_4)+\cos\tfrac{1}{2}(\theta_1+\theta_4-\theta_2-\theta_3)$$

$$= \sin(\theta_1+\theta_3)+\sin(\theta_1+\theta_4)$$

$$= -\sin(\theta_1+\theta_2).$$

It follows that if the equation of the chord PQ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta = d,$$

where 2θ is written for $\theta_1+\theta_2$, then the equation of the chord RT is

$$\frac{x}{a}\sin\theta+\frac{y}{b}\cos\theta = -\frac{\sin 2\theta}{2d}.$$

Hence, if the normals at the extremities of any two chords are concurrent, the equations of these chords can be put in the form,

$$\left.\begin{aligned} &\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-d=0, \\ \text{and} \quad &\frac{x}{a}\sec\theta+\frac{y}{b}\operatorname{cosec}\theta+\frac{1}{d}=0 \end{aligned}\right\}.$$

NOTE (4). The equation of any conic passing through the feet of the normals drawn from any point to the ellipse can now be put in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=\lambda\left\{\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-d\right\}\left\{\frac{x}{a}\sec\theta+\frac{y}{b}\operatorname{cosec}\theta+\frac{1}{d}\right\} \quad \text{(I)}$$

If the point of intersection of these normals is (h, k), it has been shown (§ 29) that the eccentric angles of the feet of the normals are given by the equation

$$\frac{ah}{\cos\theta}-\frac{bk}{\sin\theta}=a^2-b^2. \quad \ldots \quad \ldots \quad \text{(II)}$$

This equation, however, also gives the eccentric angles of the points of intersection of the ellipse and the rectangular hyperbola,

$$\frac{a^2 h}{x}-\frac{b^2 k}{y}=a^2-b^2; \quad \ldots \quad \ldots \quad \text{(III)}$$

hence the feet of the normals meeting at the point (h, k) lie on the rectangular hyperbola

$$(a^2-b^2)\,xy-a^2 hy+b^2 kx=0, \quad \ldots \quad \text{(IV)}$$

which also passes through the point (h, k) and the origin.

This rectangular hyperbola must be one of the system of conics given by equation (I); since the coefficients of x^2 and y^2 are zero, clearly $\lambda = 1$ for this conic. Equation (I) then becomes

$$xy(\tan\theta+\cot\theta)+bx\left(\frac{\cos\theta}{d}-\frac{d}{\cos\theta}\right)+ay\left(\frac{\sin\theta}{d}-\frac{d}{\sin\theta}\right)=0.$$

Comparing this equation with equation (IV), since both represent the same conic, it follows that the normals at the extremities of the chords

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-d=0$$

and

$$\frac{x}{a}\sec\theta+\frac{y}{b}\operatorname{cosec}\theta+\frac{1}{d}=0$$

are concurrent, the coordinates of their point of intersection being

$$\frac{a^2-b^2}{ad}\cos\theta\,(d^2-\sin^2\theta)\,;\ \frac{b^2-a^2}{bd}\sin\theta\,(d^2-\cos^2\theta).$$

NOTE (5). *To find the equation of the evolute of an ellipse.*

The evolute is here regarded as the locus of the intersections of consecutive normals. Hence, from a point on the evolute, two of the four normals which can be drawn coincide, consequently one of the chords joining the feet of these normals is a tangent to the ellipse.

Let the equation of this tangent be

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-1=0.$$

It follows at once that the equation of the chord joining the feet of the other two normals is

$$\frac{x}{a}\sec\theta+\frac{y}{b}\operatorname{cosec}\theta+1=0,$$

and the point of intersection of the normals is

$$\frac{a^2-b^2}{a}\cos^3\theta;\ \frac{b^2-a^2}{b}\sin^3\theta.$$

For different values of θ, the locus of this point is

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}.$$

This result can also be obtained by using Note (2) of this section.

Example (i). *From O, the centre of curvature at any point on an ellipse, the other two normals OQ, OR are drawn. Find the locus of* (1) *the middle point of QR, and* (2) *the intersections of tangents at Q and R.*

Let O be the centre of curvature at the point θ; one chord joining the feet of two of the normals from O is the tangent

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-1=0.$$

Hence the equation of the other chord QR is

$$\frac{x}{a}\sec\theta + \frac{y}{b}\operatorname{cosec}\theta + 1 = 0. \quad \ldots \quad \ldots \quad \text{(I)}$$

The equation of the diameter bisecting the chord QR is

$$\frac{x}{a}\cos\theta - \frac{y}{b}\sin\theta = 0. \quad \ldots \quad \ldots \quad \ldots \quad \text{(II)}$$

Eliminate θ between equations (I) and (II), it follows that

$$\frac{x}{a} = -\sin^2\theta\cos\theta; \quad \frac{y}{b} = -\cos^2\theta\sin\theta;$$

therefore
$$\left(\frac{xy}{ab}\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^3,$$

which is the equation of the locus of the middle point of QR.

If (X, Y) is the intersection of tangents at Q and R, the equation of its polar
$$\frac{xX}{a^2} + \frac{yY}{b^2} = 1$$
is identical with the equation (I) of the chord QR.

Hence
$$\frac{X}{a\sec\theta} = \frac{Y}{b\operatorname{cosec}\theta} = -1.$$

The locus of the point (X, Y) is accordingly

$$\left(\frac{a}{x}\right)^2 + \left(\frac{b}{y}\right)^2 = 1.$$

Example (ii). *Normals are drawn at the extremities of a chord parallel to the tangent at the point whose eccentric angle is α; find the locus of their intersections.*

Any chord parallel to the tangent at the point α is

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha - d = 0.$$

The coordinates of the point of intersection of the normals are given by
$$ax = \frac{a^2-b^2}{d}\cos\alpha\,(d^2 - \sin^2\alpha),$$

$$by = -\frac{a^2-b^2}{d}\sin\alpha\,(d^2 - \cos^2\alpha).$$

Hence
$$ax\sin\alpha + by\cos\alpha = \frac{a^2-b^2}{d}\sin\alpha\cos\alpha\cos 2\alpha,$$

$$ax\cos\alpha + by\sin\alpha = \frac{a^2-b^2}{d}d^2\cos 2\alpha.$$

The required locus is consequently

$$2\,(ax\sin\alpha + by\cos\alpha)\,(ax\cos\alpha + by\sin\alpha) = (a^2-b^2)^2\sin 2\alpha\cos^2 2\alpha.$$

Example (iii). *PP' is a diameter of an ellipse, O the centre of curvature at P, and M, N the feet of the other two normals, which can be drawn from O to the ellipse; prove that P, P', M, N lie on a circle, and find the locus of its centre.*

If P is the point θ, the equation of MN is

$$\frac{x}{a}\sec\theta + \frac{y}{b}\operatorname{cosec}\theta + 1 = 0.$$

The equation of the diameter PP' is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\operatorname{cosec}\theta = 0.$$

These four points P, P', M, N consequently lie on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \lambda\left(\frac{x}{a}\sec\theta + \frac{y}{b}\operatorname{cosec}\theta + 1\right)\left(\frac{x}{a}\sec\theta - \frac{y}{b}\operatorname{cosec}\theta\right) = 0,$$

which is a circle when

$$\lambda = \frac{a^2-b^2}{a^2\operatorname{cosec}^2\theta + b^2\sec^2\theta}.$$

The coordinates of the centre of this circle are

$$x = -\frac{a^2-b^2}{2a}\sin^2\theta\cos\theta;$$

and

$$y = \frac{a^2-b^2}{2b}\sin\theta\cos^2\theta.$$

The elimination of θ gives the required locus, whose equation is

$$4(a^2x^2+b^2y^2)^3 = a^2b^2(a^2-b^2)^2x^2y^2.$$

§ **32.** *To find the lengths of the tangents which can be drawn from any point to an ellipse.*

The equation of the tangent at the point P, whose eccentric angle is θ, is (§ **29**)

$$\frac{x-a\cos\theta}{a\sin\theta} = \frac{y-b\sin\theta}{-b\cos\theta} = \frac{r}{CD},$$

where r is the distance of the point (x, y) from the point of contact, and CD is the semi-diameter conjugate to CP. If (x, y) is the point from which the tangents are drawn, r is consequently the required length; hence the elimination of θ from the above equations gives an equation in r, where r is the length of either tangent from the point (x, y) to the ellipse.

Now $$\frac{x}{a} = \frac{r}{CD}\sin\theta + \cos\theta,$$

$$\frac{y}{b} = -\frac{r}{CD}\cos\theta + \sin\theta;$$

hence $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = \frac{r^2}{CD^2},$$

or, $$r^2 = f.CD^2,$$

where $$f \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1.$$

Hence $$r^2 = f(a^2\sin^2\theta + b^2\cos^2\theta),$$

or, $$(r^2 - a^2 f)\tan^2\theta + (r^2 - b^2 f) = 0.$$

But the equation of the tangent may be put in the form,

$$\left(1 - \frac{y^2}{b^2}\right)\tan^2\theta - \frac{2xy}{ab}\tan\theta + 1 - \frac{x^2}{a^2} = 0.$$

Hence, eliminating $\tan\theta$, the required equation follows:

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 r^4 - 2r^2 f\left\{(x^2+y^2)f + a^2b^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)\right\}$$
$$+ f^2\left[(x-ae)^2 + y^2\right]\left[(x+ae)^2 + y^2\right] = 0.$$

The term independent of r represents the product of the squares of the distances between the point, from which the tangents are drawn, and the foci.

If OP, OQ are two tangents drawn from the point O to the ellipse, it follows that

$$OP.OQ = \frac{f}{f+1}\,OH.OS.$$

Example (i). *If OP, OQ are two tangents to an ellipse, and CP', CQ' the parallel semi-diameters, show that*

$$OP.OQ + CP'.CQ' = OS.OH.$$

Let O be the point (x, y), and $f \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$, then $OP = f^{\frac{1}{2}}\,CP'$, and $OQ = f^{\frac{1}{2}}\,CQ'$. Hence,

$$OP.OQ + CP'.CQ' = OP.OQ\left(1 + \frac{1}{f}\right)$$
$$= OH.OS.$$

This result can be readily arrived at *a priori.*

Let P, Q be the points θ_1, θ_2, therefore

$$CP'^2 = a^2(1-e^2\cos^2\theta_1) = a^2(1-e\cos\theta_1)(1+e\cos\theta_1).$$

Hence

$$CP'^2 . CQ'^2 = a^4(1-e\cos\theta_1)(1-e\cos\theta_2)(1+e\cos\theta_1)(1+e\cos\theta_2).$$

But $\cos\theta_1$, $\cos\theta_2$ are given by the equation

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1,$$

or,
$$\cos^2\theta\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) - \frac{2x}{a}\cos\theta + 1 - \frac{y^2}{b^2} = 0.$$

Hence
$$a^2(1-e\cos\theta_1)(1-e\cos\theta_2)$$
$$= (x^2+y^2-2aex+a^2e^2) \div \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$$
$$= \frac{OS^2}{f+1},$$

or,
$$CP' . CQ' = \frac{OH . OS}{f+1};$$

therefore
$$OP . OQ = \frac{f . OH . OS}{f+1}.$$

Example (ii). *From a point O (ξ, η), lying on the hyperbola $x^2-y^2 = a^2-b^2$, tangents OP, OQ are drawn to an ellipse. If CO meets PQ in R, show that*

$$OP . OQ : \xi\eta = 2OR : OC.$$

The equation of the line OC is

$$\frac{x-\xi}{\xi} = \frac{y-\eta}{\eta} = \frac{r}{CO}.$$

The line PQ, being the polar of O, is

$$\frac{x\xi}{a^2} + \frac{y\eta}{b^2} = 1.$$

Hence, if $r = OR$, it follows that

$$\frac{\xi^2}{a^2}\left(\frac{r}{CO}+1\right) + \frac{\eta^2}{b^2}\left(\frac{r}{CO}+1\right) = 1.$$

Therefore
$$\frac{OR}{OC} = \frac{f}{f+1}.$$

Now $$\xi^2-\eta^2 = a^2 e^2.$$

So
$$OP.OQ = \frac{f}{f+1}\left\{(\xi-ae)^2+\eta^2\right\}^{\frac{1}{2}}\left\{(\xi+ae)^2+\eta^2\right\}^{\frac{1}{2}}$$
$$= \frac{f}{f+1}\left\{2\xi^2-2ae\xi\right\}^{\frac{1}{2}}\left\{2\xi^2+2ae\xi\right\}^{\frac{1}{2}},$$
$$= \frac{2\xi\eta f}{f+1}.$$

Therefore $$OP.OQ : \xi\eta = 2\,OR : OC.$$

Example (iii). *If p, q are the lengths of two tangents drawn from a point on the hyperbola $\frac{x^2}{a}-\frac{y^2}{b}=a-b$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and r the central distance of the point, then*

$$pq = r^2-a^2+ab-b^2;\quad p-q = 2(a-b)\frac{pq}{ab+pq},$$

and
$$(p+q)^2 = 4pq\frac{(pq+a^2)(pq+b^2)}{(pq+ab)^2}.$$

The coordinates of any point on the given hyperbola satisfy the equations
$$\frac{x^2-a^2}{a} = \frac{y^2-b^2}{b} = \frac{r^2-a^2-b^2}{a+b} = \lambda \text{ (say)}.$$

Then :

(1) $f \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1 = \frac{r^2-a^2-b^2+ab}{ab} = \frac{\lambda(a+b)+ab}{ab}$.

(2) $x^2y^2 = ab(\lambda+a)(\lambda+b) = ab(\lambda^2+abf)$.

(3) From (1). $\lambda = \frac{ab}{a+b}(f-1)$.

Let p be the length of either of the tangents, hence, as in the general case,
$$(p^2-a^2f)\tan^2\theta+(p^2-b^2f) = 0,$$
and
$$\left(\frac{y^2}{b^2}-1\right)\tan^2\theta+\frac{2xy}{ab}\tan\theta+\frac{x^2}{a^2}-1 = 0,$$
which can also be written,
$$\lambda a\tan^2\theta+2xy\tan\theta+\lambda b = 0.$$

Eliminating $\tan\theta$,
$$4x^2y^2(p^2-a^2f)(p^2-b^2f)+\lambda^2\left\{p^2(a-b)+ab(a-b)f\right\}^2 = 0.$$

Substituting the value of x^2y^2 found above, this equation reduces to

$$p^4\{\lambda^2(a+b)^2+4a^2b^2f\}-p^2\,2abf\{\lambda^2(a+b)^2+2ab(a^2+b^2)f\}$$
$$+a^2b^2f^2\{\lambda^2(a+b)^2+4a^2b^2f\}=0.$$

Again, substituting for λ in terms of f, it follows that

$$(f+1)^2p^4-2p^2abf\left\{(f+1)^2+\frac{2(a-b)^2f}{ab}\right\}+a^2b^2f^2(f+1)^2=0.$$

Hence $\quad p^2q^2=a^2b^2f^2$, or, $pq=abf=r^2-a^2+ab-b^2$,

$$p^2+q^2=2abf+\frac{4(a-b)^2f^2}{(f+1)^2},$$

therefore $\quad p-q=2(a-b)\dfrac{f}{f+1}=2(a-b)\dfrac{pq}{pq+ab},$

and, further, $\quad (p+q)^2=4abf+4(a-b)^2\dfrac{f^2}{(f+1)^2}$

$$=\frac{4f(af+b)(bf+a)}{(f+1)^2}$$

$$=\frac{4pq(pq+a^2)(pq+b^2)}{(pq+ab)^2}.$$

§ **33.** *The intersections of the ellipse and a circle.*

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0.$$

The eccentric angles of the points of intersection of this circle and the ellipse are given by the equation

$$a^2\cos^2\theta+b^2\sin^2\theta+2ga\cos\theta+2fb\sin\theta+c=0,$$

for this equation represents the condition that the point $(a\cos\theta,\ b\sin\theta)$ should lie on the circle.

Writing $t\equiv\tan\dfrac{\theta}{2}$, this equation reduces to

$$a^2(1-t^2)^2+4b^2t^2+2ga(1-t^4)+4fbt(1+t^2)+c(1+t^2)^2=0.$$

The coefficients of t and t^3 in this equation are equal; hence, if θ_1, θ_2, θ_3, θ_4 are the eccentric angles of the four points of intersection of the ellipse and circle,

$$\Sigma\tan\frac{\theta}{2}=\Sigma\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}.$$

But this is also the condition that

$$\tan \tfrac{1}{2}(\theta_1+\theta_2+\theta_3+\theta_4) = 0,$$

hence
$$\theta_1+\theta_2+\theta_3+\theta_4 = 2n\pi.$$

This condition is independent of the coefficients of the equation of the circle, consequently the necessary and sufficient condition that any four points on an ellipse should be concyclic is
$$\Sigma\,\theta = 2n\pi.$$

NOTE (1). Referring to the equation of the chord joining two points on an ellipse, it is evident that the condition $\Sigma\theta = 2n\pi$ implies that the common chords of a circle and an ellipse are equally inclined in pairs to the axis in opposite directions. (Cf. § **26.**)

NOTE (2). If a system of circles touches an ellipse at any given point, their common chords are in a fixed direction.

NOTE (3). If the circle is the circle of curvature at the point θ, since this circle intersects the ellipse in three coincident points, three of the values of θ given by the equation for their intersections must be equal. Let the fourth point of intersection be θ_1, then

$$\theta_1 + 3\theta = 2n\pi,$$

i.e. the circle of curvature at the point θ cuts the curve again at the point
$$2n\pi - 3\theta.$$

NOTE (4). If the equation

$$a^2\cos^2\theta + b^2\sin^2\theta + 2ga\cos\theta + 2fb\sin\theta + c = 0$$

be converted into equations in $\cos\theta$ and $\sin\theta$ successively, the first two terms in each are,

$$(a^2-b^2)^2\cos^4\theta + 4ga\,(a^2-b^2)\cos^3\theta + \ldots$$
$$(b^2-a^2)^2\sin^4\theta + 4fb\,(b^2-a^2)\sin^3\theta + \ldots$$

Hence
$$\Sigma\cos\theta = -\frac{4ga}{a^2-b^2}$$

and
$$\Sigma\sin\theta = -\frac{4fa}{b^2-a^2}.$$

Consequently, when the circle is the circle of curvature at the point θ,

$$-g=\frac{a^2-b^2}{4a}(3\cos\theta+\cos 3\theta)=\frac{a^2-b^2}{a}\cos^3\theta,$$

$$-f=\frac{b^2-a^2}{4b}(3\sin\theta-\sin 3\theta)=\frac{b^2-a^2}{b}\sin^3\theta,$$

which gives the coordinates of the centre of curvature at the point θ.

Example. *The three points Q, Q', Q'' on an ellipse are such that their three circles of curvature intersect at another point P on the ellipse; prove that the area of the triangle Q, Q', Q'' is $\frac{3\sqrt{3}}{4\pi}$ times the area of the ellipse.*

Let P be the point θ, the eccentric angles of the points Q, Q', Q'' are of the form $\frac{2n\pi-\theta}{3}$, or, practically, $\frac{-\theta}{3}$; $\frac{2\pi-\theta}{3}$; $\frac{4\pi-\theta}{3}$.

Now the area of the triangle $QQ'Q''$

$$=\tfrac{1}{2}ab\begin{vmatrix}\cos\frac{\theta}{3} & -\sin\frac{\theta}{3} & 1\\ \cos\frac{2\pi-\theta}{3} & \sin\frac{2\pi-\theta}{3} & 1\\ \cos\frac{4\pi-\theta}{3} & \sin\frac{4\pi-\theta}{3} & 1\end{vmatrix}$$

$$=2ab\sin\frac{\pi}{3}\sin\frac{2\pi}{3}\begin{vmatrix}\cos\frac{\theta}{3} & -\sin\frac{\theta}{3} & 1\\ \sin\frac{\theta-\pi}{3} & \cos\frac{\theta-\pi}{3} & 0\\ \sin\frac{\theta-2\pi}{3} & \cos\frac{\theta-2\pi}{3} & 0\end{vmatrix}$$

$$=2ab\sin^2\frac{\pi}{3}\sin\frac{2\pi}{3}$$

$$=\frac{3\sqrt{3}ab}{4}.$$

§ **34.** *The use of coordinates involving imaginary quantities.*

The coordinates of any point on the ellipse can be expressed in terms of the single variable θ thus:

$$x = \frac{a}{2}(e^{\iota\theta}+e^{-\iota\theta}), \quad y = \frac{b}{2\iota}(e^{\iota\theta}-e^{-\iota\theta}),$$

where $\iota \equiv \sqrt{-1}$.

If this system of coordinates is used, all the results in the previous sections relating to the normal and circle respectively can be deduced from one equation, whereas it has been found necessary with the usual coordinate system to express these equations in terms of different trigonometrical functions of the eccentric angle $\left(\cos\theta, \sin\theta, \tan\frac{\theta}{2}\right)$ to obtain particular results. The following notes will illustrate this point:—

1. To find the condition that the straight line $px+qy=r$ should touch the ellipse.

This straight line meets the ellipse at points whose eccentric angles are given by the equation

$$(ap-\iota qb)\,e^{\iota 2\theta}-2r\,e^{\iota\theta}+ap+\iota qb=0.$$

If the straight line is a tangent, this equation must have equal roots, hence

$$(ap-\iota qb)(ap+\iota qb)=a^2p^2+b^2q^2=r^2.$$

N.B. The equation of the tangent can be written

$$\left(\frac{x}{a}-\frac{\iota y}{b}\right)e^{\iota 2\theta}-2e^{\iota\theta}+\frac{x}{a}+\frac{\iota y}{b}=0.$$

2. The equation of the normal in the present notation is

$$(a^2-b^2)\,e^{\iota 4\theta}-2\,(ax-\iota by)\,e^{\iota 3\theta}+2\,e^{\iota\theta}\,(ax+\iota by)-(a^2-b^2)=0.$$

This equation, conversely, gives the eccentric angles of the feet of the normals which can be drawn from any point (x, y) to the ellipse. Hence, if the normals at the points $\theta_1, \theta_2, \theta_3, \theta_4$ are concurrent, these values of θ satisfy the above equation for some value of x and y.

NOTE (1). The coefficients of the first and last terms are equal and opposite, hence

$$e^{\iota(\theta_1+\theta_2+\theta_3+\theta_4)} = -1,$$

or,
$$\theta_1+\theta_2+\theta_3+\theta_4 = (2n+1)\pi.$$

NOTE (2). The coefficient of $e^{\iota 2\theta}$ is zero, hence

$$\sum_1^4 e^{\iota(\theta_1+\theta_2)} = 0,$$

therefore
$$\Sigma\cos(\theta_1+\theta_2) = \Sigma\sin(\theta_1+\theta_2) = 0\,;$$

or, since
$$e^{\iota\theta_1} = -e^{-\iota(\theta_2+\theta_3+\theta_4)},$$

it follows at once that

$$\sum_2^4 \left\{e^{\iota(\theta_2+\theta_3)} - e^{-\iota(\theta_2+\theta_3)}\right\} = 0,$$

or,
$$\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_4)+\sin(\theta_4+\theta_2) = 0.$$

NOTE (3). Again,

$$\Sigma e^{\iota\theta} = \frac{2(ax-\iota by)}{a^2-b^2}.$$

Hence the point of intersection of the normals at the points θ_1, θ_2, θ_3, θ_4 is given by

$$x = \frac{a^2-b^2}{2a}\Sigma\cos\theta\,;\quad y = \frac{b^2-a^2}{2b}\Sigma\sin\theta\,;$$

equating real and imaginary parts on each side of the equation.

NOTE (4). The reader may find the point of intersection of consecutive normals from the above conditions.

3. The eccentric angles of the points of intersection of the circle whose equation is

$$x^2+y^2+2gx+2fy+c = 0,$$

and the ellipse, are given by

$$(a^2-b^2)e^{\iota 4\theta}+4(ag-\iota bf)e^{\iota 3\theta}+2(a^2+b^2+2c)e^{\iota 2\theta} \\ +4(ag+\iota bf)e^{\iota\theta}+a^2-b^2 = 0.$$

NOTE (1). The coefficients of the first and last terms are equal; hence, if the four points θ_1, θ_2, θ_3, θ_4 are concyclic,

$$e^{\iota(\theta_1+\theta_2+\theta_3+\theta_4)} = 1,$$

or,

$$\Sigma\,\theta = 2n\pi.$$

NOTE (2). If this circle is the circle of curvature at the point θ, then

$$\theta_2 = \theta_3 = \theta_4 = \theta,$$

and, hence,

$$3\theta + \theta_1 = 2n\pi.$$

From the above equation it follows that

$$-\frac{4(ag-\iota bf)}{a^2-b^2} = \Sigma\, e^{\iota\theta};$$

in the case of the circle of curvature this becomes

$$\begin{aligned}
-\frac{4(ag-\iota bf)}{a^2-b^2} &= 3e^{\iota\theta} + e^{\iota\theta_1} \\
&= 3e^{\iota\theta} + e^{-\iota 3\theta} \\
&= \tfrac{1}{2}(e^{\iota\theta}+e^{-\iota\theta})^3 - \tfrac{1}{2}(e^{\iota\theta}-e^{-\iota\theta})^3 \\
&= 4\cos^3\theta + 4\iota\sin^3\theta.
\end{aligned}$$

Therefore,

$$-g = \frac{a^2-b^2}{a}\cos^3\theta; \quad -f = \frac{b^2-a^2}{b}\sin^3\theta.$$

Again,

$$\begin{aligned}
\frac{2(a^2+b^2+2c)}{a^2-b^2} &= \Sigma\, e^{\iota(\theta_1+\theta_2)} \\
&= 3e^{\iota(\theta+\theta_1)} + 3e^{\iota 2\theta} \\
&= 3e^{-\iota 2\theta} + 3e^{\iota 2\theta} \\
&= 6\cos 2\theta.
\end{aligned}$$

Therefore,

$$2c = 3(a^2-b^2)\cos 2\theta - (a^2+b^2).$$

The equation of the circle of curvature at any point θ_1 can now be written

$$x^2+y^2-\frac{2(a^2-b^2)}{a}\cos^3\theta\, x - \frac{2(b^2-a^2)}{b}\sin^3\theta\, y$$
$$+(a^2-2b^2)\cos^2\theta + (b^2-2a^2)\sin^2\theta = 0.$$

§ **35.** The frequent appearance of the length of the semi-diameter CD, conjugate to the diameter CP, in the equations of this chapter, suggests the use of this length CD as a variable of reference. Many of the lengths and angles with which one has to deal in problems on the ellipse can be very simply expressed in terms of the length CD.

It will often be found useful to thus express various parts of a problem, in a way analogous to the use of the radius of the circumscribing circle in the trigonometry of a triangle. It can be readily shown that,

1. The eccentric angle of the point P is

$$\tan^{-1}\sqrt{\frac{CD^2-b^2}{a^2-CD^2}}.$$

2. From the equation of the normal, the intercepts made on the normal by the axes of the ellipse are

$$\frac{b}{a}\,CD, \text{ and } \frac{a}{b}\,CD \text{ respectively.}$$

3. From the equation of the tangent, if PT, PT' are the intercepts made on the tangent by the axes,

$$PT\,.\,PT' = CD^2.$$

4. The perpendicular from the centre on the tangent at P is $\dfrac{ab}{CD}$.

5. The angle which the tangent at P makes with the focal distance of P is $\sin^{-1}\left(\dfrac{b}{CD}\right)$.

The Hyperbola.

§ **36.** The equation of an hyperbola referred to its axes is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Its properties can be at once deduced from those of the ellipse by substituting $\sqrt{-1}\,b$ for b.

If this use of the imaginary quantity is objected to, the universal system of coordinates given below may be employed. The results, however, are not so symmetrical as in the case of the ellipse.

Any point on the hyperbola can be expressed in the form $(a\sec\theta,\ b\tan\theta)$.

1. The equation of the chord joining the two points θ_1, θ_2 on the hyperbola is

$$\frac{x}{a}\cos\frac{\theta_1-\theta_2}{2}-\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1+\theta_2}{2}.$$

2. The equation of the tangent to the hyperbola at the point θ is

$$\frac{x}{a}-\frac{y}{b}\sin\theta=\cos\theta.$$

3. The equation of the normal to the hyperbola at the point θ is

$$ax\sin\theta+by=(a^2+b^2)\tan\theta.$$

4. The diameter conjugate to CP does not meet the curve in real points, consequently the length of the semi-conjugate diameter CD is imaginary.

If, however, CD^2 is taken equal to $a^2\tan^2\theta+b^2\sec^2\theta$, the equations of the tangent and normal can be put in the convenient forms,

$$\frac{x-a\sec\theta}{a\tan\theta}=\frac{y-b\tan\theta}{b\sec\theta}=\frac{r}{CD},$$

and

$$\frac{x-a\sec\theta}{b\sec\theta}=\frac{y-b\tan\theta}{-a\tan\theta}=\frac{r}{CD}.$$

The method of finding these equations is exactly analogous to that of finding the corresponding equations for the ellipse and is consequently left to the reader.

Illustrative Examples.

Example (i). *If from the vertex of a conic perpendiculars are drawn to the four normals which meet at the point O, these lines will meet the conic again in four concyclic points.*

The equation of the normal at the point θ is

$$ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2. \quad \ldots \quad \ldots \quad \text{(I)}$$

The equation of the perpendicular to the normal through the point $(a, 0)$ is

$$(x-a)\, b\cos\theta + ay\sin\theta = 0.$$

This straight line meets the ellipse again in a point ϕ, given by the equation

$$\cos\theta\,(1-\cos\phi) = \sin\theta\sin\phi. \quad \ldots \quad \ldots \quad \text{(II)}$$

Hence
$$\tan\frac{\phi}{2} = \tan\theta,$$

or,
$$\frac{\phi}{2} = n\pi + \theta,$$

i. e.
$$\phi = 2n\pi + 2\theta.$$

Consequently, for the four points which can thus be found,

$$\begin{aligned}\Sigma\phi &= 2n\pi + 2\Sigma\theta \\ &= 2n\pi + 2(2m+1)\pi \\ &= 2r\pi,\end{aligned}$$

which is the condition that the points ϕ_1, ϕ_2, ϕ_3, ϕ_4 should be concyclic.

Example (ii). *To find the product of the four normals which can be drawn from a given point to the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let the given point be $O\,(x, y)$.

From the equation of the normal,

$$\frac{x - a\cos\theta}{b\cos\theta} = \frac{y - b\sin\theta}{a\sin\theta} = \frac{r}{CD},$$

it follows that the product of the normals from the point O is

$$\begin{aligned}&= \prod \frac{x-a\cos\theta}{b\cos\theta}\, CD \\ &= \frac{a^4}{b^4}\prod_1^4 (a - x\sec\theta)\sqrt{(1-e\cos\theta)(1+e\cos\theta)},\end{aligned}$$

where the four values of $\cos\theta$ are given by the equation (§ 31, I),

$$c^4\cos^4\theta - 2axc^2\cos^3\theta + (a^2x^2 + b^2y^2 - c^4)\cos^2\theta + 2axc^2\cos\theta - a^2x^2 = 0. \quad \text{(I)}$$

The left-hand side of this equation is therefore identically equal to

$$c^4\prod_1^4(\cos\theta - \cos\theta_1).$$

Hence

$$c^4\prod_1^4(1-e\cos\theta) = c^4-2c^2aex+(a^2x^2+b^2y^2-c^4)e^2+2c^2axe^2-a^2x^2e^4,$$

since $$c^2 \equiv a^2-b^2 = a^2e^2.$$

Hence $$c^4\prod_1^4(1-e\cos\theta) = a^2e^2(1-e^2)(x^2+y^2-2aex+a^2e^2)$$
$$= a^2e^2(1-e^2)\,OS^2.$$

Therefore $$\Pi\sqrt{1-e\cos\theta} = \frac{b}{a^2e}OS,$$

and, similarly, $$\Pi\sqrt{1+e\cos\theta} = \frac{b\,OH}{a^2e}.$$

Again, writing equation (I) in terms of $\sec\theta$,

$$a^2x^2\Pi(a-x\sec\theta) = a^6x^2-2c^2a^4x^2-a^2x^2(a^2x^2+b^2y^2-c^4)+2c^2a^2x^4-c^4x^4;$$

therefore

$$\Pi(a-x\sec\theta) = -x^2(a^2-2a^2e^2+a^2e^4)-b^2y^2+a^4(1-2e^2+e^4),$$
$$= -b^4\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right),$$
$$\equiv -b^4f.$$

Hence the product of the normals

$$= \frac{b^2}{e^2}f\,OH\,.\,OS.$$

Further (§ 32), since the product of the tangents from O to the ellipse

$$= f\,OH\,.\,OS \div \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right),$$

the ratio of the product of the normals to the product of the tangents from the point (x, y) to an ellipse is

$$\frac{a^2b^2(f+1)}{a^2-b^2}.$$

N.B. If $\sqrt{-1}\,b$ is substituted for b, the corresponding result for the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1 \text{ is } \frac{a^2b^2}{a^2+b^2}\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right),$$

which represents the product of the perpendiculars from the point (x, y) on the asymptotes $$\frac{x}{a}\pm\frac{y}{b} = 0.$$

Example (iii). *QQ' is a chord of an ellipse parallel to one of the equi-conjugate diameters; show that the locus of the centre of the circle QCQ' for different positions of QQ' is an hyperbola.*

If θ_1 and θ_2 are the eccentric angles of the points Q, Q', since the chord QQ' is parallel to one of the lines $\frac{x}{a} \pm \frac{y}{b} = 0$, it follows that

$$\theta_1 + \theta_2 = \pm \frac{\pi}{2}.$$

The equation of any circle through the centre of the ellipse is

$$x^2 + y^2 + 2gx + 2fy = 0.$$

This circle meets the ellipse in points whose eccentric angles are given by the equation

$$a^2 \cos^2 \theta + b^2 \sin^2 \theta + 2ga \cos \theta + 2fb \sin \theta = 0.$$

If two of the values of θ given by this equation correspond to the points Q, Q', then the equation is satisfied by the values θ_1 and $\pm \frac{\pi}{2} - \theta_1$.

Hence $\quad a^2 \cos^2 \theta_1 + b^2 \sin^2 \theta_1 + 2ga \cos \theta_1 + 2fb \sin \theta_1 = 0,$

and $\quad a^2 \sin^2 \theta_1 + b^2 \cos^2 \theta_1 \pm 2ga \sin \theta_1 \pm 2fb \cos \theta_1 = 0.$

By addition and subtraction,

$$a^2 + b^2 = -2(ga \pm fb)(\cos \theta_1 \pm \sin \theta_1),$$

and $\quad (a^2 - b^2)(\cos^2 \theta_1 - \sin^2 \theta_1) = -2(ga \mp fb)(\cos \theta_1 \mp \sin \theta_1),$

i.e. $\quad (a^2 - b^2)(\cos \theta_1 \pm \sin \theta_1) = -2(ga \mp fb).$

Hence $\quad 4(g^2 a^2 - f^2 b^2) = a^4 - b^4,$

or the locus of the centre of the circle $(-g, -f)$, is the hyperbola,

$$4(a^2 x^2 - b^2 y^2) = a^4 - b^4.$$

Example (iv). *A chord of an ellipse, whose semi-axes are a, b, subtends angles $2\theta_1$, $2\theta_2$ at the foci and touches a similar, similarly situated and concentric ellipse, whose semi-axes are $a \cos \beta$, $b \cos \beta$. Prove that*

$$b(\cot \theta_1 + \cot \theta_2) = 2a \cot \beta.$$

Let the point O (h, x) be the pole of the given chord, hence the straight line

$$\frac{xh}{a^2} + \frac{yx}{b^2} = 1$$

touches the ellipse $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \beta;$

therefore $\quad \frac{h^2}{a^2} + \frac{k^2}{b^2} = \sec^2 \beta = 1 + \tan^2 \beta,$

$$\therefore \quad \tan^2 \beta = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1 \equiv f.$$

Again, a triangle whose sides are OH, OS, and AA' has angles equal to θ_1, θ_2, and ϕ, where ϕ is the angle between the tangents from the point O to the ellipse.

Hence $$\frac{\sin\theta_1}{OH} = \frac{\sin\theta_2}{OS} = \frac{\sin\phi}{2a} = \frac{\sin(\theta_1+\theta_2)}{2a}.$$

Therefore $$\cot\theta_1 + \cot\theta_2 = \frac{\sin(\theta_1+\theta_2)}{\sin\theta_1\sin\theta_2} = \frac{4a^2}{OH\,OS\sin\phi}.$$

Now $$\tan\phi = 2abf^{\frac{1}{2}} \div (a^2+b^2-x^2-y^2);$$

but
$$\begin{aligned} &4a^2b^2f + (a^2+b^2-x^2-y^2)^2 \\ &= (x^2+y^2+a^2-b^2)^2 - 4(a^2-b^2)x^2 \\ &= (x^2+y^2+a^2e^2)^2 - 4a^2e^2x^2 \\ &= OH^2\,OS^2; \end{aligned}$$

therefore
$$\begin{aligned} b(\cot\theta_1+\cot\theta_2) &= \frac{4a^2b}{OH\,OS\sin\phi} \\ &= \frac{2a}{f^{\frac{1}{2}}} = 2a\cot\beta. \end{aligned}$$

CHAPTER V

THE HYPERBOLA REFERRED TO ITS ASYMPTOTES

§ **37.** The asymptotes of an hyperbola are the two finite straight lines which touch the curve at points at infinity; the straight line at infinity is the chord of contact of these tangents.

If the asymptotes are taken as coordinate axes, since the equation of the straight line at infinity is

$$0 \,.\, x + 0 \,.\, y + c = 0,$$

the equation of the hyperbola is of the form

$$xy = c^2,$$

i. e. a conic having double contact with the straight lines $x = 0$, and $y = 0$, the line $c = 0$ being the chord of contact. In the particular case of the rectangular hyperbola the coordinate axes are rectangular.

This form of the equation to an hyperbola is useful in problems dealing with descriptive properties of the curve, and in the case of the rectangular hyperbola with metrical properties also.

§ **38.** The coordinates of any point on the hyperbola

$$xy = c^2$$

satisfy the equations $$\frac{x}{\lambda} = \frac{y}{\lambda^{-1}} = \frac{c}{1},$$

i. e. $$x = c\lambda,\; y = \frac{c}{\lambda},$$

for the point $\left(c\lambda, \dfrac{c}{\lambda}\right)$ satisfies the equation of the curve for all values of the parameter λ, and further, for different values of the parameter, these coordinates can have any value, consequently any point on the curve can be so represented.

§ **39.** *The intersections of any straight line and the hyperbola.*

Let the equation of any straight line be

$$Ax+By+c=0;$$

the parameters of the points in which this straight line meets the hyperbola are given by the equation

$$A\lambda^2+\lambda+B=0,$$

obtained by substituting $x=c\lambda$, $y=\frac{c}{\lambda}$ in the equation of the straight line.

If λ_1, λ_2 are the two roots of this equation in λ, the points of intersection are $\left(c\lambda_1, \frac{c}{\lambda_1}\right)$, $\left(c\lambda_2, \frac{c}{\lambda_2}\right)$.

It follows immediately from the above equation that

$$\lambda_1+\lambda_2=-\frac{1}{A}$$

and $$\lambda_1\lambda_2=\frac{B}{A};$$

therefore $$\frac{A}{1}=\frac{B}{\lambda_1\lambda_2}=\frac{1}{-(\lambda_1+\lambda_2)}.$$

Hence the equation of the chord joining the two points on the hyperbola, whose parameters are λ_1 and λ_2, is

$$x+\lambda_1\lambda_2 y=c(\lambda_1+\lambda_2). \quad \ldots \quad \ldots \quad \text{(I)}$$

This equation can also be written in the form

$$\frac{x}{c(\lambda_1+\lambda_2)}+\frac{y}{c(\lambda_1^{-1}+\lambda_2^{-1})}=1;$$

consequently if (X, Y) is the middle point of any chord, the equation of the chord is

$$\frac{x}{X}+\frac{y}{Y}=2. \quad \ldots \quad \ldots \quad \text{(II)}$$

NOTE (1). The parameters λ_1, λ_2 of the extremities of any one of a system of parallel chords of the hyperbola are connected by the relation $\lambda_1\lambda_2=$ constant; or, if (X, Y) be the middle point of any such chord, $\frac{Y}{X}=$ constant.

Hence the locus of the middle points of a system of chords of the hyperbola parallel to the straight lines (I) or (II) is a straight line,

$$y = kx, \text{ or, } \frac{x}{X} - \frac{y}{Y} = 0,$$

which passes through the origin of coordinates, which point is also the centre of the hyperbola.

This result can be otherwise stated thus:

The middle points of all chords of the hyperbola, which are parallel to the diameter

$$\frac{x}{X} + \frac{y}{Y} = 0,$$

lie on the diameter

$$\frac{x}{X} - \frac{y}{Y} = 0.$$

The symmetry of the result shows that the converse proposition is also true, i. e. the middle of points of chords parallel to the second diameter lie on the first.

These diameters are conjugate diameters, hence the equations of any pair of conjugate diameters are of the form,

$$x + d^2 y = 0, \text{ and } x - d^2 y = 0.$$

It should be noted that the diameter $x + d^2 y = 0$ does not meet the hyperbola in real points.

Since the equations of the asymptotes are $x = 0$ and $y = 0$, it follows that the asymptotes and any pair of conjugate diameters form an harmonic pencil.

NOTE (2). In the case of the rectangular hyperbola, the axes of coordinates being at right angles, the geometrical meaning of the parameter λ is simple. The straight line joining the origin to the point whose parameter is λ, is

$$y = \frac{x}{\lambda^2},$$

i. e. $\cot^{-1}\lambda^2$ and $\tan^{-1}\lambda^2$ are the inclinations of this radius to the asymptotes respectively.

Example. *If four points are taken on a rectangular hyperbola such that the chord joining any two is perpendicular to the chord joining the other two, and if α, β, γ, δ be the inclinations to either asymptote of the straight lines joining these points to the centre, then*

$$\tan\alpha\tan\beta\tan\gamma\tan\delta = 1.$$

Let the parameters of the four points be λ_1, λ_2, λ_3, λ_4. The equations of one pair of chords joining them are

$$x+\lambda_1\lambda_2 y = c(\lambda_1+\lambda_2),$$
$$x+\lambda_3\lambda_4 y = c(\lambda_3+\lambda_4).$$

Since these chords are by hypothesis perpendicular

$$\lambda_1.\lambda_2\lambda_3\lambda_4 = -1.$$

Hence $$\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 = 1,$$

or, $$\lambda_1^{-2}\lambda_2^{-2}\lambda_3^{-2}\lambda_4^{-2} = 1,$$

i. e. $$\tan\alpha\tan\beta\tan\gamma\tan\delta = 1.$$

The symmetry of the result shows that if one pair of chords joining the four points are perpendicular, the other two pairs of chords are also perpendicular; hence, if λ_1, λ_2, λ_3 are any three points on the rectangular hyperbola, and the line through λ_1 perpendicular to the chord joining the points λ_2, λ_3 meets the curve at the point λ_4, then the line through either of the points λ_2 or λ_3 perpendicular to the join of the other two will meet the curve in this same point λ_4, hence the orthocentre of any triangle inscribed in a rectangular hyperbola lies on the curve, and if λ_1, λ_2, λ_3 are the parameters of the vertices of this triangle, the orthocentre is the point $\left(-\dfrac{c}{\lambda_1\lambda_2\lambda_3},\ -c\lambda_1\lambda_2\lambda_3\right)$.

§ 40. *Conjugate hyperbolas.*

The hyperbolas whose equations are

$$xy = c^2, \qquad \ldots \qquad \ldots \qquad \text{(I)}$$

and $$xy = -c^2, \qquad \ldots \qquad \ldots \qquad \text{(II)}$$

respectively, are said to be conjugate; they clearly have the same asymptotes and centre.

The coordinates of any point on the first have the same sign, those of any point on the second opposite signs; hence the first curve lies entirely within the first and third

quadrants into which the common asymptotes divide the plane of coordinates, the second entirely in the second and fourth; consequently any straight line through the centre which meets one of the hyperbolas in real points meets the other only in imaginary points. These hyperbolas are connected by many interesting relations which will make the meaning of the term conjugate clear.

NOTE. The coordinates of any point on the hyperbola

$$xy = -c^2$$

can be represented by $\left(c\lambda, \; -\dfrac{c}{\lambda}\right)$, and it is to be understood in the following pages that the coordinates of any point on this hyperbola corresponding to the parameter λ, are of opposite sign *.

With this notation the equation of the chord joining any two points λ_1, λ_2 on the hyperbola is

$$x - \lambda_1 \lambda_2 \, y = c\,(\lambda_1 + \lambda_2).$$

It can also be shown, as in § 39, that the equations of any pair of conjugate diameters are of the form

$$x - d^2 y = 0\,; \quad x + d^2 y = 0\,;$$

and, consequently, conjugate diameters of any hyperbola are also conjugate diameters of the conjugate hyperbola; in particular they have the same axes, the major axis of the one being the minor axis of the other, and vice versa.

Suppose the diameter $x - d^2 y = 0$ meets the hyperbola $xy - c^2 = 0$ in the real points P, P', the parameters of these points are $\pm d$.

The conjugate diameter $x + d^2 y = 0$ meets the conjugate hyperbola $xy = -c^2$ in two real points D, D', whose parameters are also $\pm d$.

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* Results symmetrical with those of the hyperbola $xy = c^2$ can be obtained by using $\left(\lambda\sqrt{-c^2}, \; \dfrac{1}{\lambda}\sqrt{-c^2}\right)$ as the coordinates of any point on the conjugate hyperbola.

Let ω be the angle between the asymptotes, it follows that

$$CP^2 = CP'^2 = c^2 d^2 + \frac{c^2}{d^2} + 2c^2 \cos\omega,$$

and $$CD^2 = CD'^2 = c^2 d^2 + \frac{c^2}{d^2} - 2c^2 \cos\omega;$$

hence $$CP^2 - CD^2 = 4c^2 \cos\omega = \text{constant}.$$

NOTE (1). If the hyperbolas are rectangular $CP^2 = CD^2$.

NOTE (2). The expression for the length of the radius CP can be written

$$CP^2 = \left(cd - \frac{c}{d}\right)^2 + 4c^2 \cos^2 \tfrac{1}{2}\omega.$$

If CP is a minimum, i.e. if it is the semi-major axis of the hyperbola, then $d = \pm 1$.

Hence the equation of the major axis is

$$x - y = 0,$$

and its length is $4c \cos \frac{1}{2}\omega$.

Similarly it may be shown that the equation of the minor axis is $$x + y = 0,$$

and its length $4c \sin \frac{1}{2}\omega$.

Hence, if a and b are the lengths of the major and minor semi-axes, $c^2 = \frac{1}{4}(a^2 + b^2)$, and $\omega = 2\tan^{-1}\frac{b}{a}$.

NOTE (3). The coordinates of the points P, P', D, D' are

$$\left(\pm cd, \ \pm\frac{c}{d}\right) \text{ and } \left(\pm cd, \ \mp\frac{c}{d}\right),$$

i.e. $PP'DD'$ is a parallelogram whose sides are parallel to the asymptotes, and whose lengths are $2cd$ and $\frac{2c}{d}$ respectively, its area is $2c^2 \sin\omega$, which is constant.

§ **41.** *The tangent.*

The equation of the chord joining any two points on the hyperbola whose parameters are λ_1, λ_2 is

$$x + \lambda_1 \lambda_2 y = c(\lambda_1 + \lambda_2).$$

If the points λ_1, λ_2 coincide, this equation becomes that of the tangent at the point; hence the equation of the tangent at the point λ is

$$x+\lambda^2 y = 2c\lambda.$$

NOTE (1). The equation of the tangent at the point λ on the conjugate hyperbola $xy+c^2=0$ is

$$x-\lambda^2 y = 2c\lambda.$$

NOTE (2). The intercepts made by the tangent at the point λ on the asymptotes are $2c\lambda$ and $\dfrac{2c}{\lambda}$;

hence (i) the intercept made by the asymptotes on any tangent is bisected at the point of contact,

(ii) the area of the triangle formed by the asymptotes and any tangent to the curve is constant and equal to

$$2c^2 \sin\omega.$$

Example (i). *Find the locus of the intersections of two straight lines which are perpendicular to each other and are tangents respectively to a rectangular hyperbola and its conjugate.*

Let the straight lines be

$$x+\lambda^2 y-2c\lambda = 0,$$
$$x-\mu^2 y-2c\mu = 0.$$

Since they are perpendicular

$$\lambda^2\mu^2 = 1, \qquad \text{i.e. } \lambda\mu = \pm 1.$$

Hence the locus required is that of the intersections of the straight lines,

$$\lambda^2 y-2c\lambda+x = 0,$$
$$\lambda^2 x \mp 2c\lambda - y = 0.$$

By cross-multiplication,

$$\frac{\lambda^2}{2c(y\pm x)} = \frac{\lambda}{x^2+y^2} = \frac{1}{2c(x\mp y)},$$

or, the locus is $\qquad (x^2+y^2)^2 = \pm 4c^2(x^2-y^2).$

Example (ii). *Tangents at the points of intersection of* $xy=c^2$ *and the ellipse* $x^2+y^2 = c^2(\cos\omega+\sec\omega)$ *are perpendicular to the asymptotes.*

If the parameter of either point of intersection is λ, since the point $\left(c\lambda, \dfrac{c}{\lambda}\right)$ is also on the ellipse,

$$\lambda^4+1 = \lambda^2(\cos\omega+\sec\omega),$$

i.e. $$\lambda^2\cos\omega-1 = 0, \text{ or, } \lambda^2-\cos\omega = 0,$$

which are the required conditions that the tangents at these points should be perpendicular to the asymptotes.

The equation of the tangent at the point λ on the hyperbola, $$\lambda^2 y-2c\lambda+x = 0,$$

may also be regarded as a quadratic equation in the variable λ, giving the parameters of the points of contact of the tangents which can be drawn from the point (x, y) to the curve.

Hence, if the tangents from the point (x, y) meet the curve at the points λ_1, λ_2, then

$$\lambda_1+\lambda_2 = \frac{2c}{y}, \text{ and } \lambda_1\lambda_2 = \frac{x}{y};$$

or, $$x = \frac{2c}{\lambda_1^{-1}+\lambda_2^{-1}}, \text{ and } y = \frac{2c}{\lambda_1+\lambda_2}.$$

NOTE (1). The tangents at the extremities of a chord whose middle point is (X, Y) intersect at the point

$$\left(\frac{c^2}{Y}, \frac{c^2}{X}\right).$$

NOTE (2). The tangents to the hyperbola at the points λ_1, λ_2 are at right angles if

$$\lambda_1^2\lambda_2^2-(\lambda_1^2+\lambda_2^2)\cos\omega+1 = 0.$$

If (x, y) is the point of intersection of these tangents, it follows that

$$x^2+y^2+2xy\cos\omega = 4c^2\cos\omega.$$

This equation represents the locus of the intersections of pairs of orthogonal tangents, and is called the director circle; its centre is the centre of the hyperbola and its radius is

$$(a^2-b^2)^{\frac{1}{2}}.$$

It is interesting to note that in the case of the rectangular hyperbola this locus reduces to the point-circle

$$x^2+y^2=0,$$

i.e. the only real orthogonal tangents to a rectangular hyperbola are the asymptotes.

NOTE (3). The tangents from any point (x, y) to the hyperbola are real, coincident, or imaginary, according as c^2-xy is positive, zero, or negative.

Hence the point (x, y) is outside, on, or inside the curve, according as xy is $<$, $=$, or $>c^2$.

NOTE (4). Since the parameters of the points of contact of tangents from the point (X, Y) to the hyperbola are given by

$$\lambda_1+\lambda_2=\frac{2c}{Y}; \text{ and } \lambda_1\lambda_2=\frac{X}{Y},$$

the equation of the chord of contact (i.e. the chord joining the points λ_1, λ_2) is

$$xY+yX=2c^2.$$

This equation consequently represents the polar of the point (X, Y) with respect to the hyperbola.

Example (i). *Find the locus of the intersections of pairs of tangents to an hyperbola length which cut off a constant length on one of the asymptotes.*

If λ_1, λ_2 are the points of contact of one pair of tangents the intercepts made by the tangents on the asymptote $y=0$ are $2c\lambda_1$, $2c\lambda_2$.

The given condition is, accordingly,

$$\lambda_1-\lambda_2=\text{constant}.$$

Now if (x, y) is the point of intersection of tangents, λ_1, λ_2 are given by

$$\lambda^2y-2c\lambda+x=0;$$

therefore $$\frac{4c^2}{y^2}-\frac{4x}{y}=(\lambda_1-\lambda_2)^2=\text{constant},$$

i.e. the required locus is $\quad c^2-xy=ky^2.$

Example (ii). *To find the locus of the intersection of tangents to a rectangular hyperbola which contain a given angle θ.*

If λ_1, λ_2 are the points of contact of two of the tangents, then

$$\tan\theta = \frac{\lambda_1^2-\lambda_2^2}{1+\lambda_1^2\lambda_2^2}.$$

But if (x, y) is the corresponding point on the locus, λ_1, λ_2 are given by

$$\lambda^2 y - 2c\lambda + x = 0;$$

therefore

$$\tan\theta = \frac{4c\sqrt{c^2-xy}}{x^2+y^2},$$

and the required locus is

$$(x^2+y^2)^2 \tan^2\theta = 16c^2(c^2-xy).$$

§ **42.** *The normal to a rectangular hyperbola.*

The equation of the normal to an hyperbola whose asymptotes are not rectangular involves the angle between the asymptotes; this equation is

$$\lambda(1-\lambda^2\cos\omega)\,y+\lambda(\cos\omega-\lambda^2)\,x+(\lambda^4-1)\,c = 0.$$

Problems requiring the equation of the normal are better treated by taking the equation of the hyperbola in the form discussed in the last chapter.

The tangent to a rectangular hyperbola whose equation is

$$xy = c^2;$$

at the point $\left(c\lambda, \dfrac{c}{\lambda}\right)$ is $\dfrac{x-c\lambda}{\lambda^2} = \dfrac{y-\dfrac{c}{\lambda}}{-1}$,

hence the equation of the normal at the same point is

$$\frac{x-c\lambda}{1} = \frac{y-\dfrac{c}{\lambda}}{\lambda^2},$$

which may be written

$$c\lambda^4 - x\lambda^3 + y\lambda - c = 0.$$

This equation, being the condition that the normal at the point λ should pass through the point (x, y), gives the parameters of the feet of the normals which can be drawn

from the point (x, y) to the hyperbola. This equation is quartic in the variable λ, hence four normals can be drawn from any point to a rectangular hyperbola.

If the parameters of the feet of these normals are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, they must satisfy the conditions

$$\Sigma\, \lambda_1 \lambda_2 = 0 \text{ and } \lambda_1 \lambda_2 \lambda_3 \lambda_4 = -1,$$

which may otherwise be written

$$\Sigma\, \lambda_1 \lambda_2 = \Sigma \frac{1}{\lambda_1 \lambda_2} = 0.$$

These two conditions are independent of the coordinates x and y; consequently they are the conditions which must be satisfied by the parameters of any four points on the hyperbola the normals at which are concurrent.

If the Cartesian coordinates of four such points are (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), these conditions become

$$\Sigma\, x_1 x_2 = \Sigma\, y_1 y_2 = 0.$$

If these conditions are satisfied, the coordinates of the point of intersection of the normals are

$$x = c\, \Sigma \lambda\, ; \;\; y = -c\, \Sigma\, \lambda_1 \lambda_2 \lambda_3 = c\, \Sigma \frac{1}{\lambda},$$

or, expressed in the Cartesian coordinates of the points they are $\quad (x_1 + x_2 + x_3 + x_4)$, and $(y_1 + y_2 + y_3 + y_4)$.

NOTE (I). The four normals, which can be drawn from any point $\left(c\mu, \dfrac{c}{\mu}\right)$ on the rectangular hyperbola to the curve, meet the curve in points whose parameters are given by the equation

$$c\lambda^4 - c\lambda^3\mu + \frac{c\lambda}{\mu} - c = 0,$$

which reduces to $\quad (\lambda^3\mu + 1)(\lambda - \mu) = 0.$

The value $\lambda = \mu$ corresponds to the normal at the chosen point; the parameters of the feet of the other normals are given by

$$\lambda^3 = -\frac{1}{\mu}.$$

Hence, two of these are imaginary, and the third is the point

$$\left(-\frac{c}{\sqrt[3]{\mu}},\ -c\sqrt[3]{\mu}\right).$$

Hence, conversely, the normal to the hyperbola at the point λ meets the curve again at the point $-\frac{1}{\lambda^3}$.

NOTE (2). To find the conditions that the normals at the points of intersection of any two straight lines and a rectangular hyperbola should be concurrent, and to find the coordinates of the point of intersection of these normals.

Let the two straight lines be

$$Ax+By=c,$$
$$A'x+B'y=c.$$

If these lines meet the rectangular hyperbola at the points λ_1, λ_2; λ_3, λ_4 respectively, comparing the equations of the lines with those of the corresponding chords, it follows that

$$\lambda_1\lambda_2=\frac{B}{A},\qquad \lambda_3\lambda_4=\frac{B'}{A'};$$
$$\lambda_1+\lambda_2=\frac{1}{A},\quad \lambda_3+\lambda_4=\frac{1}{A'}.$$

The conditions that the four normals at these points should be concurrent are

$$\lambda_1\lambda_2\lambda_3\lambda_4=-1 \quad\text{and}\quad \lambda_1\lambda_2+\lambda_3\lambda_4+(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)=0,$$

hence $\qquad AA'+BB'=0,$

and $\qquad AB'+A'B=-1,$

which conditions are equivalent to

$$\frac{A'}{B}=\frac{B'}{-A}=\frac{1}{A^2-B^2}.$$

The coordinates of the point of intersection of the normals are $x=c\,\Sigma\lambda=c\left(\frac{1}{A}+\frac{1}{A'}\right)$; $y=c\,\Sigma\frac{1}{\lambda}=c\left(\frac{1}{B}+\frac{1}{B'}\right)$.

Consequently, if the normals at the extremities of two chords of a rectangular hyperbola are concurrent, the chords are at right angles and their equations are of the form

$$\begin{cases} Ax+By = c \\ Bx-Ay = c\,(A^2-B^2). \end{cases}$$

Example (i). *The locus of the intersections of normals to a rectangular hyperbola at the extremities of a system of parallel chords is another rectangular hyperbola.*

Let
$$Ax+By = c$$
be any one of the system of parallel chords, and suppose

$$\frac{A}{B} = \tan\gamma.$$

The chord joining the feet of the other two normals, which can be drawn to the curve from the point of intersection of the normals at the end of this chord, is

$$Bx-Ay = c\,(A^2-B^2).$$

The coordinates of the point of intersection of the normals are

$$x = c\left(\frac{1}{A}+\frac{A^2-B^2}{B}\right) = c\left(\frac{1}{A}+A\tan\gamma-A\cot\gamma\right),$$

$$y = c\left(\frac{1}{B}-\frac{A^2-B^2}{A}\right) = c\left(\frac{1}{A}\tan\gamma-A+A\cot^2\gamma\right);$$

therefore
$$x\cot\gamma+y = \frac{c}{A}(\cot\gamma+\tan\gamma),$$

$$x-y\cot\gamma = cA(\tan\gamma-\cot^3\gamma),$$

hence the required locus is

$$(x\cot\gamma+y)(x-y\cot\gamma) = c^2\operatorname{cosec}^4\gamma\,(1-\tan^2\gamma).$$

Example (ii). *The four normals are drawn from any point* (x, y) *to the rectangular hyperbola. If the tangents at the feet of the normals meet in* (ξ_r, η_r), $r = 1, 2\ldots 6$, *show that*

$$\sum_{r=1}^{r=6}\left(\frac{\xi_r}{\eta_r}\right) = \sum_{r=1}^{r=6}\left(\frac{\eta_r}{\xi_r}\right) = 0,$$

and
$$\xi_1\,\xi_2\ldots\xi_6+\eta_1\,\eta_2\ldots\eta_6 = 0.$$

The parameters of the feet of these normals are given by

$$c\lambda^4-x\lambda^3+y\lambda-c = 0.$$

The points of contact of tangents from the point (ξ, η) are given by

$$\eta\lambda^2-2c\lambda+\xi = 0.$$

Therefore $$\frac{\xi}{\eta} = \lambda_1 \lambda_2.$$

Hence $$\Sigma \frac{\xi}{\eta} = \Sigma \lambda_1 \lambda_2 = 0,$$

and $$\Sigma \frac{\eta}{\xi} = \Sigma \frac{1}{\lambda_1 \lambda_2} = 0.$$

Again $$\frac{\xi_1 \xi_2 \ldots \xi_6}{\eta_1 \eta_2 \ldots \eta_6} = (\lambda_1 \lambda_2 \lambda_3 \lambda_4)^3 = (-1)^3 = -1.$$

§ 43. *The intersections of an hyperbola and a circle.*

The equation of any circle referred to the asymptotes of the hyperbola as axes of coordinates is

$$x^2 + y^2 + 2xy \cos \omega + 2gx + 2fy + d = 0.$$

The parameters of the points of intersection of this circle and the hyperbola are obtained by substituting $x = c\lambda$ and $y = \frac{c}{\lambda}$ in this equation, hence

$$c^2 \lambda^4 + 2gc\lambda^3 + (d + 2c^2 \cos \omega)\lambda^2 + 2fc\lambda + c^2 = 0.$$

If λ_1, λ_2, λ_3, λ_4 are the values of λ given by this equation, since the coefficients of the first and last terms are equal,

$$\lambda_1 \lambda_2 \lambda_3 \lambda_4 = 1.$$

This is the necessary condition that the four points whose parameters are λ_1, λ_2, λ_3, λ_4 should lie on a circle; it is also sufficient, since any three points lie on a circle. When the hyperbola is rectangular the coordinates of the centre of this circle are $(-g, -f)$, i.e. in terms of the parameters of the points

$$\left(\frac{c}{2} \Sigma \lambda, \quad \frac{c}{2} \Sigma \frac{1}{\lambda}\right).$$

These results can otherwise be stated thus:

If the four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) on the hyperbola $xy - c^2 = 0$ are concyclic, the product of their abscissae is c^4, and the centre of the circle is the point

$$\tfrac{1}{2}(x_1 + x_2 + x_3 + x_4), \quad \tfrac{1}{2}(y_1 + y_2 + y_3 + y_4).$$

Note (1). The condition that four points on the hyperbola should be concyclic is also the condition that the common chords of the circle and hyperbola should be equally inclined in pairs to either axis of the hyperbola.

NOTE (2). If this circle is the circle of curvature at the point λ, then three of the points of intersection of the circle and hyperbola coincide with λ. Hence, if the fourth point of intersection is λ_1, then $\lambda_1 \lambda^3 = 1$,

i. e. the circle of curvature at the point $\left(c\lambda, \frac{c}{\lambda}\right)$ meets the hyperbola again at the point $\left(\frac{c}{\lambda^3}, c\lambda^3\right)$. The equation of the chord of curvature at the point λ is accordingly

$$x + \frac{y}{\lambda^2} = c\left(\lambda + \frac{1}{\lambda^3}\right),$$

or,
$$\lambda^3 x + \lambda y = c(\lambda^4 + 1).$$

Incidentally it may be noted that four chords of curvature meet at any point.

The equation to the circle of curvature may now be found,

for
$$-2g = c\Sigma\lambda = c(3\lambda + \lambda_1) = c\left(3\lambda + \frac{1}{\lambda^3}\right),$$

$$-2f = c\Sigma\frac{1}{\lambda} = c\left(\lambda^3 + \frac{3}{\lambda}\right);$$

$$d + 2c^2\cos\omega = c^2\Sigma\lambda_1\lambda_2 = c^2\left(3\lambda^2 + \frac{3}{\lambda^2}\right);$$

hence the equation is

$$x^2 + y^2 + 2xy\cos\omega - cx\left(3\lambda + \frac{1}{\lambda^3}\right) - cy\left(\frac{3}{\lambda} + \lambda^3\right)$$
$$+ c^2\left(3\lambda^2 + \frac{3}{\lambda^2} - 2\cos\omega\right) = 0.$$

When the hyperbola is rectangular this equation reduces to

$$x^2 + y^2 - cx\left(3\lambda + \frac{1}{\lambda^3}\right) - cy\left(\frac{3}{\lambda} + \lambda^3\right) + c^2\left(3\lambda^2 + \frac{3}{\lambda^2}\right) = 0.$$

(*a*) Since this equation is of the sixth degree in λ, six circles of curvature can be drawn to meet at any point (x, y); the parameters of the points at which they touch the hyperbola are given by this equation, and these parameters satisfy four independent conditions.

(*b*) The centre of curvature of any point λ on the rectangular hyperbola is $\frac{c}{2}\left(3\lambda+\frac{1}{\lambda^3}\right),\ \frac{c}{2}\left(\frac{3}{\lambda}+\lambda^3\right)$.

(*c*) The locus of the centre of curvature, i. e. the equation of the evolute, may be found as follows:

Since $$x=\frac{c}{2}\left(3\lambda+\frac{1}{\lambda^3}\right),\ y=\frac{c}{2}\left(\frac{3}{\lambda}+\lambda^3\right),$$

therefore $$x+y=\frac{c}{2}\left(\lambda+\frac{1}{\lambda}\right)^3,$$

$$x-y=\frac{c}{2}\left(\frac{1}{\lambda}-\lambda\right)^3.$$

Hence $$(x+y)^{\frac{2}{3}}-(x-y)^{\frac{2}{3}}=(4c)^{\frac{2}{3}}.$$

(*d*) The radius of curvature ρ, at the point λ, is given by

$$\rho^2=g^2+f^2-d$$

$$=\frac{c^2}{4}\left(3\lambda+\frac{1}{\lambda^3}\right)^2+\frac{c^2}{4}\left(\lambda^3+\frac{3}{\lambda}\right)^2-3c^2\left(\frac{1}{\lambda^2}+\lambda^2\right)$$

$$=\frac{c^2}{4}\left(\lambda^2+\frac{1}{\lambda^2}\right)^3,$$

or, $$\rho=\frac{c}{2}\left(\lambda^2+\frac{1}{\lambda^2}\right)^{\frac{3}{2}}.$$

Illustrative Examples.

Example (i). *To find the sum of the squares of the lengths of the normals which can be drawn from the point* (x, y) *to a rectangular hyperbola.*

The required sum

$$=\Sigma(x-c\lambda)^2+\Sigma\left(y-\frac{c}{\lambda}\right)^2,$$

where the four values of λ are given by the equation

$$c\lambda^4-\lambda^3x+\lambda y-c=0.$$

Hence $$\Sigma\lambda_1\lambda_2=0,\ \text{and}\ \Sigma\lambda^2=(\Sigma\lambda)^2.$$

$$\text{Sum} \quad = 4x^2+4y^2-2cx\Sigma\lambda-2cy\Sigma\frac{1}{\lambda}+c^2\Sigma\lambda^2+c^2\Sigma\frac{1}{\lambda^2},$$
$$= 3(x^2+y^2);$$

i. e. the sum of the squares of the normals from any point is equal to three times the square of the distance of that point from the centre.

Example (ii). *If the circle circumscribing the triangle formed by the tangents at the points (x_1, y_1), (x_2, y_2), (x_3, y_3) on a rectangular hyperbola passes through the centre of the curve, then*

$$\frac{x_1+x_2+x_3}{x_1x_2x_3}+\frac{y_1+y_2+y_3}{y_1y_2y_3}=0,$$

and the centre of the circle is

$$-\frac{y_1y_2y_3}{c^2}, \ -\frac{x_1x_2x_3}{c^2}.$$

Let the parameters of the points of contact of the tangents be λ_1, λ_2, λ_3; and write $s_1 \equiv \Sigma\lambda$, $s_2 \equiv \Sigma\lambda_1\lambda_2$, $s_3 \equiv \lambda_1\lambda_2\lambda_3$.

The coordinates of the vertices of the circumscribing triangle are

$$\frac{2cs_3}{\lambda(s_1-\lambda)}; \ \frac{2c}{s_1-\lambda},$$

where λ has either of the values λ_1, λ_2, or λ_3.

A circle whose centre is (g, f), and which passes through the origin, is
$$x^2+y^2-2gx-2fy=0.$$

If any one of the vertices of the triangle lies on this circle, then
$$f\lambda^3-(fs_1-gs_3-c)\lambda^2-gs_1s_3\lambda+cs_3^2=0.$$

But this condition is satisfied when λ is equal to λ_1, λ_2, or λ_3, hence λ_1, λ_2, λ_3 are the roots of this equation in λ.

It follows that

(1) $s_3=-c\dfrac{s_3^2}{f}$, i. e. $f=-cs_3$.

(2) $s_2=-gs_1\dfrac{s_3}{f}$, i. e. $g=\dfrac{cs_2}{s_1}$.

(3) $s_1=s_1-\dfrac{gs_3}{f}-\dfrac{c}{f}$, i. e. $g=-\dfrac{c}{s_3}$.

Hence the centre of the circle is $\left(-\dfrac{c}{s_3}, \ -cs_3\right)$, with the condition $s_2s_3+s_1=0$.

These results are identical with those required.

Example (iii). *A_1, A_2, A_3, A_4 are the feet of the normals drawn to an ellipse from a point P, and B_1 is the fourth point where the circle $A_2\,A_3\,A_4$ cuts the rectangular hyperbola $A_1\,A_2\,A_3\,A_4$; the points B_2, B_3, B_4 being determined in the same manner. Prove that the points B_1, B_2, B_3, B_4 lie on an invariable ellipse, whose centre is at P and whose axes are parallel to those of the original ellipse.*

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the points A_1, A_2, A_3, A_4 be $(x_1\,y_1)$, $(x_2\,y_2)$, $(x_3\,y_3)$, $(x_4\,y_4)$, the point P be (h, k), and B_1 (ξ_1, η_1).

The rectangular hyperbola is (p. 66, IV)

$$(a^2-b^2)\,xy-a^2hy+b^2kx=0,$$

which may be written

$$\left(x-\frac{ha^2}{a^2-b^2}\right)\left(y+\frac{kb^2}{a^2-b^2}\right)=-\frac{hk\,a^2\,b^2}{(a^2-b^2)^2},$$

or,
$$(x-\alpha)\,(y+\beta)=c^2.$$

The coordinates of any point on this hyperbola can be expressed in terms of the parameter λ, thus:

$$x=\alpha+c\lambda,\quad y=-\beta+\frac{c}{\lambda}.$$

The points A_1, A_2, A_3, A_4 are consequently given by

$$\frac{1}{a^2}(\alpha+c\lambda)^2+\frac{1}{b^2}\left(\beta-\frac{c}{\lambda}\right)^2=1,$$

in which equation the coefficients of λ^4 and λ^0 are in the ratio $b^2:a^2$; hence, if λ_1, λ_2, λ_3, λ_4 are the parameters of the A-points,

$$\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4=\frac{a^2}{b^2}.$$

Now if μ_1 is the parameter of B_1, since the points μ, λ_2, λ_3, λ_4 lie on a circle,

$$\mu_1\lambda_2\lambda_3\lambda_4=1,$$

or,
$$\mu_1=\frac{b^2\lambda_1}{a^2}.$$

Therefore, if
$$x_1=\alpha+c\lambda_1,$$

then
$$\xi_1=\alpha+c\,\frac{b^2}{a^2}\lambda_1;$$

therefore
$$\frac{a^2}{b^2}(\xi_1-h)=x_1.$$

Similarly $$\frac{b^2}{a^2}(\eta_1-k)=y_1.$$

But the point (x_1, y_1) lies on the ellipse, hence any B-point (ξ, η) lies on the ellipse $$\frac{a^4}{b^4}(\xi-h)^2+\frac{b^2}{a^4}(\eta-k)^2=1,$$
which satisfies the required conditions.

Example (iv). *To find the equation of a parabola which has 4-point contact with the hyperbola $xy-c^2=0$ at any point.*

Let the equation of the parabola be
$$(x+by)^2+2gx+2fy+d=0.$$

The parameters of the points of intersection of this parabola and the hyperbola are given by
$$\left(c\lambda+\frac{bc}{\lambda}\right)^2+2gc\lambda+2f\frac{c}{\lambda}+d=0,$$
or, $$c^2\lambda^4+2gc\lambda^3+(d+2bc^2)\lambda^2+2fc\lambda+b^2c^2=0.$$

If the parabola has 4-point contact at the point λ, all the roots of this equation are equal, hence
$$-2g=4c\lambda,\quad d+2bc^2=6c^2\lambda^2,$$
$$-2f=4c\lambda^3,\quad b^2=\lambda^4,\quad \text{i.e. } b=\pm\lambda^2.$$

The equation of the parabola is, accordingly,
$$(x+\lambda^2y)^2-4c\lambda x-4c\lambda^3y+4c^2\lambda^2=0,$$
or, $$(x-\lambda^2y)^2-4c\lambda x-4c\lambda^3y+8c^2\lambda^2=0.$$

The former equation represents a pair of coincident straight lines (p. 17, Note (2)), hence one and only one parabola can be drawn having 4-point contact with the hyperbola at a given point.

Example (v). *The circle of curvature at any point P of the rectangular hyperbola $xy=c^2$ touches the tangent to the hyperbola at the point Q. If the straight line PQ is a tangent to the hyperbola $4xyk=(k+1)^2c^2$, prove that P is one of the points where the two curves*
$$(3+k)x^4-(1+3k)y^4=6c^4(k-1),$$
$$(3k+1)x^4-(k+3)y^4=6c^4(1-k)$$
meet the first hyperbola.

Let P be the point λ, and Q the point μ.

The chord $$x+\lambda\mu y=c(\lambda+\mu)$$
touches the hyperbola $$4kxy=(k+1)^2c^2.$$

Hence $\qquad k(\lambda+\mu)^2 = (k+1)^2\lambda\mu,$ or, $\mu = k\lambda,$ or, $\dfrac{\lambda}{k}.$

The tangent at the point $k\lambda$ is

$$x+k^2\lambda^2 y = 2k\lambda c.$$

Expressing that the square of the perpendicular on this straight line from the centre of curvature at P, $\left\{\frac{c}{2}\left(3\lambda+\frac{1}{\lambda^3}\right),\ \frac{c}{2}\left(\frac{3}{\lambda}+\lambda^3\right)\right\}$, is equal to the square of the radius of curvature at P, viz.

$$\frac{c^2}{4}\left(\lambda^2+\frac{1}{\lambda^2}\right)^3,$$

it follows that

$$[k^2(3\lambda^4+\lambda^8)-4\lambda^4 k+3\lambda^4+1]^2 = (1+k^4\lambda^4)(\lambda^4+1)^3.$$

This equation is quartic in k, and gives the four values of k for points $k\lambda$ on the hyperbola, the tangents at which touch the circle of curvature at P. But this circle touches the curve in three points at P, hence three values of k given by this equation will be equal to 1, since the tangent at these three coincident points touch the circle of curvature at P.

The coefficients of k^4 and k^0 are in the ratio $3\lambda^8+6\lambda^4-1:3+6\lambda^4-\lambda^8$.

Hence $$1^3.\,k = \frac{3+6\lambda^4-\lambda^8}{3\lambda^8+6\lambda^4-1},$$

or, the equation for λ is

$$\lambda^8(3k+1)+6\lambda^4(k-1)-(k+3) = 0,$$

i.e. the point $\left(c\lambda,\ \dfrac{c}{\lambda}\right)$ lies on the curve

$$(3k+1)x^4-(k+3)y^4 = 6(1-k).$$

The second curve follows at once from this by writing $\dfrac{1}{k}$ for k throughout.

CHAPTER VI

CUBIC CURVES

§ **44.** A cubic curve is represented by an equation of the third degree; the most general equation of the third degree, when any point is denoted by the three coordinates x, y, and z, contains ten terms, viz.: x^3, y^3, z^3, x^2y, x^2z, y^2x, y^2z, z^2x, z^2y, xyz with ten corresponding coefficients.

These ten coefficients are equivalent to nine independent ratios; hence, in general, nine conditions (e. g. nine points on the curve) determine a cubic. It follows that in general through any eight points any number of cubic curves can be drawn. The case of a conic section may be compared to this; five points determine a conic section, and through four points any number of conics can in general be drawn.

§ **45.** The coordinates of any point on a cubic curve can in a large number of cases be expressed in terms of a single variable or parameter. The properties of such curves can be investigated by simple algebraical processes, analogous to those discussed in the last three chapters. The present chapter deals mainly with this class of cubic curve; such curves are called Unicursal Cubics. It is proposed first to discuss shortly those properties of a cubic which occur in a general treatment of them, then to explain how the equations of cubics can be reduced to certain standard forms, giving illustrative examples of those cubics which most frequently occur. In the general discussion, three coordinates, connected by a constant relation, will be used, and the application to special systems, such as the Cartesian, will be illustrated in the examples.

Suppose that the coordinates of any point on a cubic curve can be determined by the equations

$$\frac{x}{f_1(\lambda)} = \frac{y}{f_2(\lambda)} = \frac{z}{f_3(\lambda)},$$

where f_1, f_2, f_3 are functions of the single variable λ.

If the variable λ is eliminated from these equations, the eliminant will be the equation of the corresponding cubic curve.

In Cartesian coordinates the corresponding equations are

$$x = F_1(\lambda); \; y = F_2(\lambda).$$

§ 46. *The intersections of a straight line and a cubic curve.*

Let the equation of any straight line be

$$Ax + By + Cz = 0.$$

The equation $Af_1(\lambda) + Bf_2(\lambda) + Cf_3(\lambda) = 0$,

obtained by substituting the values of the coordinates of any point on the curve in terms of the parameter λ in the equation of the line, gives the values of the parameter λ at the points in which this straight line meets the cubic. Any straight line meets a cubic in three points, for to find the coordinates of these points we have to solve the linear equation of the line and the cubic equation of the curve simultaneously; hence this equation in λ must be of the third degree, and it is thus clear that the functions f_1, f_2, f_3 are at most of the third degree in the parameter, and one at least must be of the third degree.

The following particular cases in respect to the roots of this equation in λ may arise:

1. Since every cubic equation has one real root, every straight line meets a cubic in at least one real point; consequently a cubic cannot be a closed curve.

2. Two of the values of λ may be such that on substituting their values in the functions f_1, f_2, f_3 the same values of the coordinates $x : y : z$ are obtained.

In this case the straight line meets the curve at a point where two distinct branches meet; such a point is called a double point, the two values of λ corresponding to the same point on the two different branches of the curve.

3. If two of the values of λ are equal, then

(*a*) The straight line is a tangent to the curve at the corresponding point, the third value of λ corresponds to the point in which the tangent meets the curve, other than the point of contact. This value of λ must be real, hence every tangent to a cubic meets the curve again in a real point.

Or, (*b*) the line passes through a point on the curve at which two branches meet, the value of the parameter λ corresponding to each branch of the curve being the same. Such a point is called *a cusp*. The line meets the curve at one other point corresponding to the third value of λ given by the equation; this value must be real, hence every straight line passing through a cusp on the curve meets the curve again in a real point.

Example. *To find the points of intersection of the straight line* $3x-2y-z=0$ *and the cubic*

$$\frac{x}{\lambda^3}=\frac{y}{(1-\lambda)^3}=\frac{z}{-1}.$$

The parameters of these points of intersection are given by

$$3\lambda^3-2(1-\lambda)^3+1=0,$$

i.e. $$5\lambda^3-6\lambda^2+6\lambda-1=0,$$

or, $$(5\lambda-1)(\lambda^2-\lambda+1)=0.$$

Hence $$\lambda=\frac{1}{5},\ +\omega,\ -\omega^2,\ \text{where } \omega=\sqrt[3]{-1}.$$

The last two values of λ when substituted in the equations of the curve each give $x:y:z::1:1:1$; hence the line meets the curve at the double point $(1:1:1)$, at which two branches of the curve cross each other, and at the point whose parameter is $\frac{1}{5}$.

4. If all three values of λ are equal, then

(*a*) The straight line is a tangent to the curve at this point, and also cuts the curve at the same point. Such a point is called *a point of inflexion* on the curve.

If λ is the value of the parameter given in this case, there are three equations giving the values of the coefficients of the equation in terms of λ; but the coefficients of this equation are functions of the coefficients in the equation of the given line, i.e. of two independent quantities only. These coefficients can consequently be eliminated, the eliminant being an equation in λ; the values of λ given by this latter equation will evidently correspond to the points of inflexion on the cubic.

Or, (*b*) the line is the tangent to the curve at a cusp.

Example (i). *Prove that the inflexions at a finite distance on the curve given by the equations*

$$x = \frac{t^2(t+1)}{t-a};\ y = \frac{t(t+1)^2}{t-a}$$

correspond to the values of t which satisfy the equation

$$t^3 - 3at^2 - 3at - a = 0,$$

and that they lie on the straight line

$$x(2+3a) - y(1+3a) + 1 = 0.$$

Let the equation of any straight line be

$$Ax + By + 1 = 0.$$

The parameters of the points of intersection of this straight line and the cubic are given by

$$At^2(t+1) + Bt(t+1)^2 + t - a = 0,$$

or, $$(A+B)t^3 + (A+2B)t^2 + (B+1)t - a = 0.$$

Suppose all the roots of this equation are equal to t.

Then $$1+B = \frac{3a}{t};\ A+2B = -\frac{3a}{t^2};\ \text{and}\ A+B = \frac{a}{t^3}.$$

Therefore $$-B = \frac{a}{t^3} + \frac{3a}{t^2} = 1 - \frac{3a}{t},$$

i.e. $$t^3 - 3at^2 - 3at - a = 0.$$

The points given by this equation are consequently points of inflexion.

These points of inflexion will lie on a straight line $Ax+By+1 = 0$, if the equation giving their parameters is identical with that giving the parameters of the points of intersection of this line and the cubic; i. e. if
$$\frac{A+B}{1} = \frac{A+2B}{-3a} = \frac{1+B}{-3a} = 1.$$

These conditions give
$$A+B = 1;\quad A+2B = -3a;\quad 1+B = -3a;$$
which are simultaneously satisfied by $A = 2+3a$, and $B = -(1+3a)$.

Example (ii). *Find the point of inflexion on the cubic*
$$(2x+y-3a)^3 = (x+y-2a)^2\,(y-4a).$$

Any point on this cubic is given by the equations
$$\frac{2x+y-3a}{\lambda^2} = \frac{x+y-2a}{\lambda^3} = \frac{y-4a}{1}.$$

The equation of any straight line is
$$A(2x+y-3a)+B(x+y-2a)+C(y-4a) = 0,$$
and the intersections of this straight line and the cubic are given consequently by the equation
$$A\lambda^2+B\lambda^3+C = 0.$$

Since the coefficient of λ is zero, it follows that if the roots of this equation are λ_1, λ_2, λ_3,
$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0 \text{ unless } C \text{ is zero.}$$

Hence, if all the roots are equal, their value is infinite.

If the chosen straight line passes through this point ($\lambda = \infty$), B must be zero, and the equation giving the other points of intersection becomes $\frac{C}{\lambda^2} + A = 0$. It follows that in general a straight line through this point meets the curve only once at this point (the line $y-4a = 0$ cuts it in three coincident points), hence this point is a point of inflexion on the curve; its coordinates are given by $2x+y-3a = 0$ and $y-4a = 0$; hence the point of inflexion is $(-\frac{1}{2}a, 4a)$.

If, however, C is zero, the given straight line passes through the point $\lambda = 0$, and the equation giving the other points of intersection reduces to

$$B\lambda^2 + A\lambda = 0,$$

consequently a second value of λ must also be zero, i.e. every straight line through this point meets the curve in two coincident points, the parameters of which have the same value; hence this point is a cusp, its coordinates are given by $2x+y-3a = 0$ and $x+y-2a = 0$; i.e. the point (a, a) is a cusp.

§ 47. The singular points on a cubic curve can now be defined and distinguished as follows:—

(i) A point of inflexion is one at which the curve crosses the tangent; the tangent meets the curve in three coincident points, but any other straight line through the point meets the curve in only one point.

(ii) Double points are of three kinds:

(*a*) A node at which two distinct branches of the curve cross; there are two real tangents to the curve at the point.

(*b*) A conjugate point, which is a node the tangents at which are imaginary.

(*c*) A cusp at which two branches of the curve meet and have one common tangent.

The tangent at a double point meets the curve in three coincident points, and any other straight line through a double point meets the curve in two coincident points; this distinguishes a double point from a point of inflexion.

§ 48. A particular case of the form of the functions f_1, f_2, f_3 should be noted.

If two of these functions f_1, f_2 have a common quadratic factor, the intersection of the corresponding lines of reference $x = 0$, $y = 0$ is a double point on the curve; this point is a node, conjugate point, or cusp, according as this quadratic factor has real, imaginary, or equal linear factors.

For suppose that the functions f_1, f_2 have the common quadratic factor

$$a\lambda^2 + b\lambda + c.$$

The parameters of the points of intersection of any straight line $Ax+By=0$ passing through the intersection of the lines of reference $x=0$ and $y=0$ are given by

$$A f_1(\lambda)+B f_2(\lambda)=0.$$

This equation is equivalent to

$$(a\lambda^2+b\lambda+c)(\lambda-d)=0,$$

hence all such straight lines meet the curve in two points corresponding to the values of λ given by the equation

$$a\lambda^2+b\lambda+c=0.$$

The coordinates of these points are the same, viz. $(0:0:1)$; consequently this is a double point on the curve.

If the third value of λ, viz. d, is equal to either of those given by the quadratic equation above, the corresponding straight line meets the curve in three coincident points, i. e. is a tangent to the curve at the double point.

Since one tangent corresponds to each root of the quadratic equation, the point is a node, conjugate point, or cusp, according as the roots of this equation are real, imaginary, or equal.

Example (i). *To find the nature of the double point on the curve*

$$x^2(y-x)=z(y-2x)^2.$$

Any point on this curve satisfies the equations

$$\frac{x}{(\lambda-1)^2}=\frac{y}{(\lambda-1)^2(\lambda+1)}=\frac{z}{\lambda}.$$

Any straight line, $Ax+By=0$,

meets the curve at points whose parameters are given by

$$(\lambda-1)^2\{A+B(\lambda+1)\}=0.$$

The common quadratic factor $(\lambda-1)^2$ gives two equal roots 1, 1; hence the point is a cusp. If the third value of λ is also 1, then $A+2B=0$, consequently the straight line $y-2x=0$ is the tangent at this cusp.

Example (ii). $(x-z)^2(z-y) = 2yz^2$.

The coordinates of any point on this curve satisfy the implicit equations
$$\frac{x}{(\lambda^2+2)(\lambda+1)} = \frac{y}{\lambda^2} = \frac{z}{\lambda^2+2}.$$

Since the common quadratic factor λ^2+2 has imaginary factors, the point $(0:1:0)$ is a conjugate point on the curve; the equation of the tangents at the point is $x^2-2xz+3z^2 = 0$.

Example (iii). *Show that the point* (a, a) *is a conjugate point on the curve* $a(x-y)^2+2y(y-a)^2 = 0$, *and find the equation of the tangents.*

The coordinates of any point on the curve satisfy the equations
$$\frac{x-a}{(\lambda^2+2)(\lambda+1)} = \frac{y}{\lambda^2} = \frac{y-a}{\lambda^2+2}.$$

Hence, since λ^2+2 is a common quadratic factor and has imaginary factors, the point $x-a = y-a = 0$ is a conjugate point on the curve.

The third point of intersection of any straight line
$$A(x-a)+B(y-a) = 0$$
with the curve is given by
$$A(\lambda+1)+B = 0,$$
and when λ has either of the values $\pm\sqrt{-2}$, the corresponding line is a tangent; their equations are accordingly
$$x-y = +\sqrt{-2}\,(y-a), \text{ and } x-y = -\sqrt{-2}\,(y-a),$$
or,
$$(x-y)^2+2(y-a)^2 = 0.$$

§ **49.** *To find the equation of a chord of any unicursal cubic, and of the tangent at any point.*

Suppose the parameters of the extremities of the chord are λ_1 and λ_2, and let $Ax+By+Cz = 0$ be the equation of the chord. Since the points λ_1 and λ_2 are on this line, it follows that
$$A\,f_1(\lambda_1)+B\,f_2(\lambda_1)+C\,f_3(\lambda_1) = 0,$$
and
$$A\,f_1(\lambda_2)+B\,f_2(\lambda_2)+C\,f_3(\lambda_2) = 0.$$

The ratios $A:B:C$ being found by cross-multiplication, the corresponding equation of the chord is
$$x\,\{f_2(\lambda_1)\,f_3(\lambda_2)-f_2(\lambda_2)\,f_3(\lambda_1)\} + \text{the } y \text{ and } z \text{ terms with similar coefficients} = 0.$$

Since each of the coefficients in this equation vanishes when $\lambda_1 = \lambda_2$, $(\lambda_1 - \lambda_2)$ is a factor of each. If this factor is divided out the equation of the chord is obtained. If in this equation λ_1 and λ_2 are each put equal to λ, the equation becomes that of the tangent at the point λ. This method applies to all special cases which should be worked out for particular problems *a priori*; in the examples in the later portions of this chapter this working is omitted to save repetition.

Example (i). *To find the tangent at any point to the curve*

$$x^2(x-a) = y^3.$$

The coordinates of any point on the curve satisfy the equations

$$\frac{x}{\lambda^3} = \frac{y}{\lambda^2} = \frac{x-a}{1}.$$

If $$Ax + By + C(x-a) = 0$$

is the equation of the chord joining the points λ_1, λ_2, then

$$A\lambda_1^3 + B\lambda_1^2 + C = 0,$$

$$A\lambda_2^3 + B\lambda_2^2 + C = 0.$$

By cross-multiplication

$$\frac{A}{\lambda_1^2 - \lambda_2^2} = \frac{B}{\lambda_2^3 - \lambda_1^3} = \frac{C}{\lambda_1^2\lambda_2^2(\lambda_1 - \lambda_2)},$$

or, $$\frac{A}{\lambda_1 + \lambda_2} = \frac{B}{-(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)} = \frac{}{\lambda_1^2\lambda_2^2}.$$

The equation of the chord joining the points λ_1, λ_2 is accordingly

$$x(\lambda_1 + \lambda_2) - y(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2) + (x-a)\lambda_1^2\lambda_2^2 = 0.$$

Putting $\lambda_1 = \lambda_2 = \lambda$, the equation of the tangent at λ becomes

$$2x - 3y\lambda + (x-a)\lambda^3 = 0,$$

or, $$x(2+\lambda^3) - 3y\lambda - a\lambda^3 = 0.$$

Example (ii). *Show that, on the curve*

$$x = \frac{at}{t^3 + bt^2 + ct + d}, \quad y = \frac{a}{t^3 + bt^2 + ct + d},$$

the condition that the tangents at the three points where the parameter has the values t_1, t_2, t_3 *may be concurrent is*

$$3(t_1t_2 + t_2t_3 + t_3t_1) + 2b(t_1 + t_2 + t_3) + b^2 = 0.$$

If $$Ax+By+a=0$$
is the equation of the chord joining the points t_1, t_2,

then $$At_1+B+t_1^3+bt_1^2+ct_1+d=0,$$
and $$At_2+B+t_2^3+bt_2^2+ct_2+d=0.$$

By cross-multiplication
$$\frac{A}{(t_2^3-t_1^3)+b(t_2^2-t_1^2)+c(t_2-t_1)}=\frac{B}{t_1t_2(t_1^2-t_2^2)+bt_1t_2(t_1-t_2)+d(t_2-t_1)}$$
$$=\frac{1}{t_1-t_2},$$
hence $$-\frac{A}{t_1^2+t_2^2+t_1t_2+b(t_1+t_2)+c}=\frac{B}{t_1t_2(t_1+t_2)+bt_1t_2-d}=\frac{1}{1}.$$

Putting $t_1=t_2=t$ for the tangent,
$$-\frac{A}{3t^2+2bt+c}=\frac{B}{2t^3+bt^2-d}=\frac{1}{1}.$$

Hence the equation of the tangent at the point t is
$$x(3t^2+2bt+c)-y(2t^3+bt^2-d)-a=0.$$

Conversely this equation gives the parameters of the points of contact of tangents from the point (x, y) to the curve. The equation can be written
$$2yt^3-(3x-by)t^2-2bxt-cx-dy+a=0.$$

If t_1, t_2, t_3 are the parameters of the points of contact,
$$t_1+t_2+t_3=\frac{3x}{2y}-\frac{b}{2},$$
and $$t_1t_2+t_2t_3+t_3t_1=-\frac{bx}{y},$$
hence $$3(t_1t_2+t_2t_3+t_3t_1)+2b(t_1+t_2+t_3)+b^2=0,$$
which, being independent of the coordinates x and y, is the condition that the tangents at the points t_1, t_2, t_3 should be concurrent.

§ 50. *The asymptotes of a cubic.*

An asymptote to any curve is a straight line which touches the curve at some point at infinity, which straight line does not lie wholly at infinity, e. g. it must cut off a finite length from one of the axes of coordinates.

An asymptote may thus meet a cubic in two or three coincident points at infinity; in the former case it meets the curve at some finite point also, in the latter the cubic has a singular point at infinity.

In Cartesian coordinates the straight line

$$Ax + By + 1 = 0$$

is an asymptote to the curve, if it meets the curve at infinity in two or three coincident points, provided that one of the coefficients A and B is finite, and they are not both zero.

If the coordinates of any point on the curve are given by

$$x = f_1(\lambda); \quad y = f_2(\lambda),$$

any value of λ which makes one or both of the coordinates infinite represents a point at infinity on the curve, the tangent at this point λ is an asymptote if its equation satisfies the above conditions.

Example (i). *To find the asymptotes of the curve*

$$4x^3 = (a + 3x)(x^2 + y^2).$$

The coordinates of any point on this curve satisfy the equations

$$x = \frac{a(1+\lambda^2)}{1-3\lambda^2}; \quad y = \frac{a\lambda(1+\lambda^2)}{1-3\lambda^2}.$$

The values of λ which make a coordinate infinite are ∞, $\pm\frac{1}{\sqrt{3}}$.

The equation of the tangent at any point λ is

$$4x(1+3\lambda^2) - 8y\lambda - (3x+a)(1+\lambda^2)^2 = 0,$$

or,

$$x(1+6\lambda^2-3\lambda^4) - 8y\lambda - a(1+\lambda^2)^2 = 0.$$

The tangents at the points $\pm\frac{1}{\sqrt{3}}$ are accordingly

$$3\sqrt{3}y = \pm(3x - 2a),$$

which are consequently asymptotes to the curve.

Neglecting all except the highest powers of λ in the equation of the tangent, it follows that $\quad 3x + a = 0$

is the equation of the tangent at the point ∞, and is the other asymptote.

Example (ii). *To find the asymptotes of the curve*

$$(x + a)y^2 = (y + b)x^2.$$

The coordinates of any point on this curve are given by

$$x = \frac{a - b\lambda^2}{\lambda - 1}; \quad y = \frac{a - b\lambda^2}{\lambda(\lambda - 1)}.$$

A coordinate is infinite when $\lambda = \infty$, 0, or 1.

The equation of the tangent at any point is

$$(x+a)(b\lambda^2-2a\lambda+a)+(y+b)(a-2b\lambda+b\lambda^2)\lambda^2+2\lambda(a-b\lambda)^2.$$

The tangents at the points ∞, 0, and 1 are

$$y+b=0;\quad x+a=0;\quad x-y=a-b,$$

which are the equations of the asymptotes.

§ **51.** *To find the equations of the cubic in certain standard forms.*

The meaning of the various forms of cubics, which occur most frequently in problems, is often better understood from more geometrical reasoning. In this article the method of expressing the equation of a cubic in its simplest forms is explained. One or two preliminary propositions are necessary.

1. A cubic can only have one double point, for if it had two the straight line joining them would meet the cubic in four points, which is impossible.

Cubics are accordingly divided into three classes, Nodal cubics, Cuspidal cubics, and Non-singular cubics, according as they have one node, one cusp, or no double point.

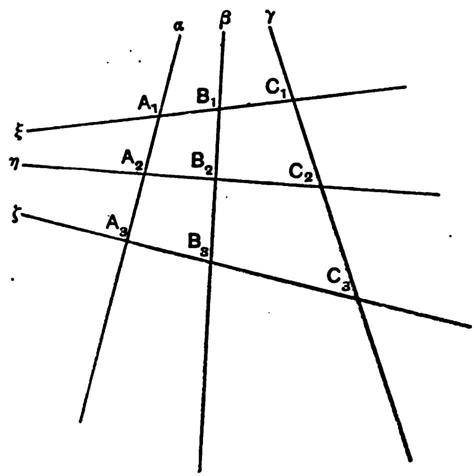
2. *Any cubic which passes through eight fixed points passes also through a ninth fixed point.*

Let $S=0$ and $S'=0$ represent two cubics passing through the eight fixed points, any other cubic through these points is represented by the equation $S-kS'=0$ for some value of k; hence any cubic through the eight points passes also through the ninth point of intersection of $S=0$ and $S'=0$.

It follows that a cubic is not determined by nine given points, if these points happen to be the intersections of two cubics.

3. Let the straight lines whose equations are $\xi=0$, and $\eta=0$, meet any cubic curve in the points A_1, B_1, C_1, and A_2, B_2, C_2 respectively; and suppose the straight lines A_1A_2 ($\alpha=0$), B_1B_2 ($\beta=0$) meet the curve again in the points A_3, B_3.

If $\zeta = 0$ is the straight line $A_3 B_3$, since $\xi\eta\zeta = 0$ and $\alpha\beta\gamma = 0$ are two cubics passing through the eight points A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, any other cubic through these eight points passes also through the ninth point of intersection of these straight lines, i.e. the lines $C_1 C_2$ ($\gamma = 0$) and $A_3 B_3$ ($\zeta = 0$) meet at a point C_3 on the curve.



It follows at once that the equation of any cubic can be put in the form

$$\xi\eta\zeta = k\,\alpha\beta\gamma,$$

where $\xi = 0$, $\eta = 0$, $\zeta = 0$, $\alpha = 0$, $\beta = 0$, $\gamma = 0$ are the equations of six straight lines.

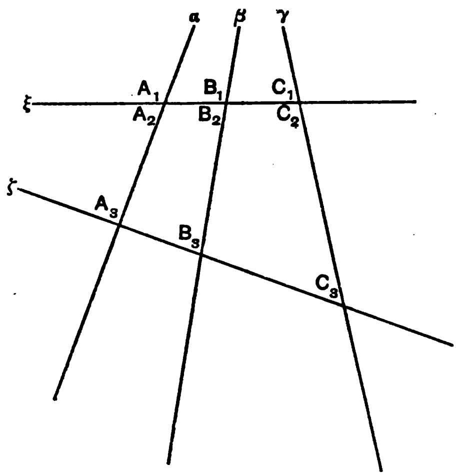
Certain particular cases are of importance; the reader may deduce several others in a manner similar to the following.

(i) Suppose the two straight lines $\xi = 0$, $\eta = 0$ coalesce. Two of the intersections of the lines α, β, and γ with the curve then coincide.

Hence the equation

$$\xi^2 \zeta = k\,\alpha\beta\gamma$$

represents a cubic, the straight lines α, β, γ being tangents to the curve at the points of intersection of the line $\xi = 0$



with the curve; the third points of intersection of these tangents and the cubic lying on the straight line $\zeta = 0$.

(ii) Suppose all the straight lines $\xi = 0$, $\eta = 0$, $\zeta = 0$ coalesce. Then each of the straight lines $\alpha = 0$, $\beta = 0$, $\gamma = 0$ meets the curve in three coincident points, and the straight line $\xi = 0$ cuts the curve at one point at each of their points of contact; hence these points are points of inflexion.

Hence the equation

$$\xi^3 = k\,\alpha\beta\gamma$$

represents a cubic having three points of inflexion; these points of inflexion lie on the straight line $\xi = 0$, and $\alpha = 0$, $\beta = 0$, $\gamma = 0$ are the inflexional tangents.

NOTE. The equation of the straight line on which the points of inflexion lie can be put in the form

$$l\alpha+m\beta+n\gamma=0,$$

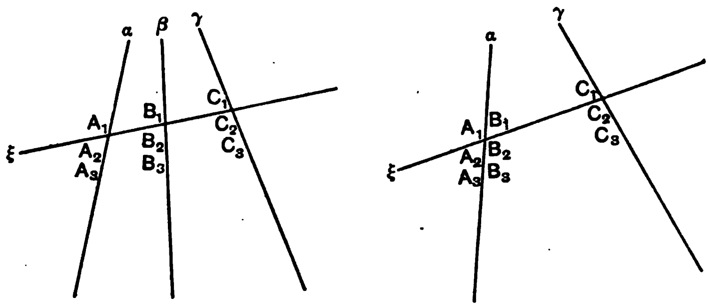
or, by choosing a suitable system of coordinates,

$$\alpha+\beta+\gamma=0;$$

hence the equation

$$(x+y+z)^3=k\,xyz$$

represents a cubic having three points of inflexion, which points lie on the line $x+y+z=0$, and $x=0$, $y=0$, $z=0$ are the inflexional tangents.



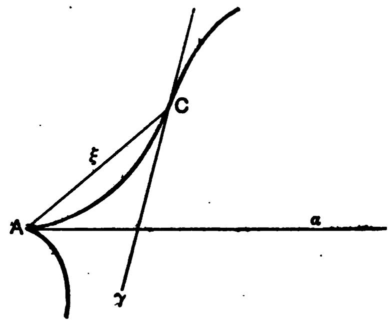
(iii) Suppose the straight lines $\xi=0$, $\eta=0$, $\zeta=0$ coalesce, and also the two lines $\alpha=0$, $\beta=0$ become one.

The straight lines $\alpha=0$, $\gamma=0$ each meet the curve in three coincident points; these lines are tangents and their points of contact lie on the straight line $\xi=0$; but this straight line meets the curve in two coincident points at the first point of contact, and in one at the second, consequently the equation

$$\xi^3=k\,\alpha^2\gamma$$

represents a cuspidal cubic, the straight line $\alpha=0$ being

the tangent at the cusp, $\gamma = 0$ an inflexional tangent, and $\xi = 0$ the join of the cusp and point of inflexion.



In particular, the cubic

$$y^3 = a^2 x$$

has a cusp at infinity, the origin is a point of inflexion at which $x = 0$ is a tangent. This curve is called a cubical parabola.

The cubic

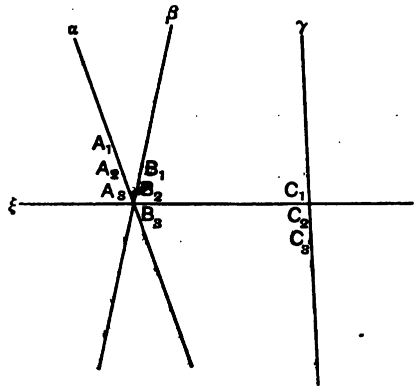
$$y^3 = ax^2$$

has a point of inflexion at infinity, the origin is a cusp, the axis $x = 0$ being the cuspidal tangent. This cubic is called a semi-cubical parabola.

(iv) Suppose, as before, that the lines $\xi = 0$, $\eta = 0$, $\zeta = 0$ coalesce, and also that the lines $\alpha = 0$, $\beta = 0$ intersect on the line $\xi = 0$.

Then again, the three lines $\alpha = 0$, $\beta = 0$, $\gamma = 0$ meet the curve in three coincident points, i.e. are tangents to the curve at singular points; these points lie on the straight line $\xi = 0$, and this straight line cuts the curve at two coincident points at the intersection of $\alpha = 0$ and $\beta = 0$, and in one point on $\gamma = 0$.

Hence the equation $\xi^3 = k\,\alpha\beta\gamma,$
where the point $\alpha = 0$, $\beta = 0$ lies on $\xi = 0$, represents a nodal cubic, $\alpha = 0$ and $\beta = 0$ are tangents at the node,



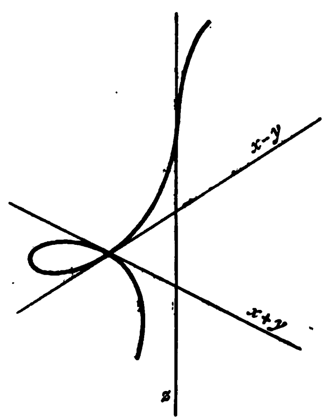
$\gamma = 0$ is an inflexional tangent, $\xi = 0$ is the join of the node and point of inflexion.

Since the lines ξ, α, β are concurrent, their equations can be put in the forms $x = 0$, $x+y = 0$, $x-y = 0$.

Hence the equation

$$x^3 = z(x^2-y^2)$$

represents a nodal cubic, the point $x = y = 0$ being the node, the straight lines $x^2-y^2 = 0$ the tangents at the node, and the point $x = z = 0$ being a point of inflexion, the tangent at which is $z = 0$.



If the tangents at the node are imaginary, i.e. if this point is a conjugate point, the equation can be written

$$x^3 = z(x^2+y^2).$$

§ **52.** *Cuspidal cubics.*

Let the equation of any cuspidal cubic be

$$u^3 = v^2 w,$$

where $u = 0$, $v = 0$, $w = 0$ are three straight lines.

The coordinates of any point on this cubic satisfy the equations

$$\frac{u}{\lambda} = \frac{v}{1} = \frac{w}{\lambda^3},$$

hence any point on the cubic may be denoted by the parameter λ.

Since $w = \lambda^2 u$, which is satisfied by $+\lambda$ and $-\lambda$, represents a straight line through the intersection of $u = 0$ and $w = 0$, the chord joining any two points whose parameters are $\pm\lambda$ passes through the point $u = w = 0$; such points are called corresponding points on the cubic.

NOTE (1). The parameters of the points of intersection of any straight line $Au + Bv + Cw = 0$ and the cubic are given by the equation $A\lambda + B + C\lambda^3 = 0$.

If the three values of λ given by this equation are $\lambda_1, \lambda_2, \lambda_3$, since the coefficient of λ^2 is zero,

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

This is consequently the condition that any three points on the cubic whose parameters are $\lambda_1, \lambda_2, \lambda_3$ should be collinear.

In particular, if the straight line is a tangent to the curve, two of these values of λ are equal. Let the values be $\lambda, \lambda, \lambda_1$, hence $\lambda_1 + 2\lambda = 0$.

It follows that the tangent at the point λ meets the curve again at the point -2λ.

NOTE (2). The equation of the chord joining any two points λ_1, λ_2 is

$$(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)\, u - \lambda_1\lambda_2(\lambda_1 + \lambda_2)\, v - w = 0.$$

Hence the equation of the tangent at the point λ is

$$3\lambda^2 u - 2\lambda^3 v - w = 0.$$

This equation is cubic in the variable λ, consequently from any point $u:v:w$ three tangents (two of which may be imaginary) can be drawn to a cuspidal cubic, the parameters of the points of contact being given by this equation. The coefficient of λ is zero; if then $\lambda_1, \lambda_2, \lambda_3$ are the values of λ given by this equation, it follows that

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0.$$

This equation, being independent of the coordinates u, v, and w, is the condition that the tangents at the three points $\lambda_1, \lambda_2, \lambda_3$ on the cubic should be concurrent.

Several illustrative examples of cuspidal cubics, in which u, v, and w have particular values, are now given. The properties of the semi-cubical parabola, when the axes are rectangular, are worked out at greater length because the equation of the normal can be simply found and some other metrical properties examined.

Example (i). *The equation of a semi-cubical parabola is*

$$x^3 = ay^2,$$

the origin being the cusp, and the axis of x the cuspidal tangent.

The coordinates of any point on this curve satisfy the equations

$$\frac{x}{\lambda^2} = \frac{y}{\lambda^3} = \frac{a}{1},$$

or, the coordinates of any point on the curve are $(a\lambda^2, a\lambda^3)$.

NOTE (I). In the case of rectangular axes, if λ be the parameter of any point, x and y the coordinates of the same point, then

$$y = \lambda x.$$

This evidently is the equation of the straight line joining the point λ to the origin. Hence the parameter λ is the tangent of the inclination to the cuspidal tangent of the radius vector from the origin to the point. If this

inclination is θ, the coordinates of the corresponding point can be taken as $(a\tan^2\theta,\ a\tan^3\theta)$.

NOTE (2). The parameters of the intersections of any straight line $Ax+By+a=0$ and the cubic are given by

$$B\lambda^3+A\lambda^2+1=0.$$

Since the coefficient of λ is zero, the condition that the three points λ_1, λ_2, λ_3 should be collinear is

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}=0.$$

The tangent at any point λ meets the curve again at the point $-\frac{1}{2}\lambda$.

NOTE (3). The equation of the chord joining any two points λ_1, λ_2 on the curve is

$$(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)\,x-(\lambda_1+\lambda_2)\,y-\lambda_1^2\lambda_2^2a=0.$$

NOTE (4). The equation of the tangent at the point λ is consequently $$3\lambda x-2y-\lambda^3a=0.$$

Since the coefficient of λ^2 is zero, the condition that the tangents at the points λ_1, λ_2, λ_3 should be concurrent is

$$\lambda_1+\lambda_2+\lambda_3=0.$$

Example. *To find the locus of the intersections of tangents to a semi-cubical parabola which are at right angles.*

The points of contact of tangents from the point (x, y) to the curve are given by $$a\lambda^3-3x\lambda+2y=0.$$

Let λ_1, λ_2, λ_3 be the three points. The tangents at λ_1, λ_2 are at right angles; hence $$9\lambda_1\lambda_2+4=0.$$

But $\lambda_1\lambda_2\lambda_3=-\dfrac{2y}{a}$; therefore $\lambda_3=\dfrac{9y}{2a}$.

Also $$\lambda_1\lambda_2+\lambda_3(\lambda_1+\lambda_2)=-\frac{3x}{a},$$

or, $$\lambda_1\lambda_2-\lambda_3^2=-\frac{3x}{a};$$

therefore $$\frac{4}{9}+\frac{81y^2}{4a^2}=\frac{3x}{a}.$$

The required locus is consequently the parabola

$$729y^2=108ax-16a^2.$$

NOTE (5). The equation of the normal at the point λ is

$$3\lambda(y-a\lambda^3)+2(x-a\lambda^2)=0,$$

or
$$2x+3\lambda y=3a\lambda^4+2a\lambda^2.$$

This equation is of the fourth degree in λ, hence four normals can be drawn from any point (x, y) to a semi-cubical parabola.

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the parameters of the feet of these normals, it follows that

$$\Sigma\lambda=0 \text{ and } \Sigma\lambda_1\lambda_2=\frac{2}{3}.$$

These two conditions are necessary and sufficient in order that the normals at any four points $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ should be concurrent.

COR.: The condition that the normals at any three points $\lambda_1, \lambda_2, \lambda_3$ should be concurrent is

$$\Sigma\lambda^2+\Sigma\lambda_1\lambda_2+\frac{2}{3}=0,$$

which is obtained by eliminating λ_4 from the above two conditions.

Example. *To find the coordinates of the centre of curvature at the point* λ.

The centre of curvature is the intersection of two consecutive normals.

If the coordinates of this centre of curvature are (x, y), let the feet of the normals from this point be $\lambda, \lambda, \lambda_1, \lambda_2$.

Hence
$$\lambda_1+\lambda_2=-2\lambda.$$

$$2\lambda(\lambda_1+\lambda_2)+\lambda^2+\lambda_1\lambda_2=\frac{2}{3}.$$

or,
$$\lambda_1\lambda_2=\frac{2}{3}+3\lambda^2.$$

Also
$$\frac{2x}{3a}=-\lambda^2\lambda_1\lambda_2=-\lambda^2\left(\frac{2}{3}+3\lambda^2\right),$$

and $$\frac{y}{a} = \lambda^2(\lambda_1+\lambda_2)+2\lambda\lambda_1\lambda_2,$$
$$= 4\lambda\left(\lambda^2+\frac{1}{3}\right),$$

hence the centre of curvature is the point

$$-\frac{a\lambda^2}{2}(2+9\lambda^2); \quad \frac{4a\lambda}{3}(3\lambda^2+1).$$

NOTE (6). The points of intersection of the cubic and any conic $$\alpha x^2+\beta y^2+2hxy+2gx+2fy+c=0$$
are given by

$$\beta a^2\lambda^6+2ha^2\lambda^5+\alpha a^2\lambda^4+2fa\lambda^3+2ga\lambda^2+c=0.$$

Since the coefficient of λ is zero, it follows that the necessary and (since any five points lie on a conic) sufficient condition that six points on the cubic should lie on a conic is

$$\Sigma\frac{1}{\lambda}=0.$$

COR.: The corresponding equation for the circle

$$x^2+y^2+2gx+2fy+c=0$$

is $$a^2\lambda^6+a^2\lambda^4+2fa\lambda^3+2ga\lambda^2+c=0.$$

Hence, if six points lie on a circle, their parameters satisfy three conditions, viz.:

$$\Sigma\lambda=0, \quad \Sigma\lambda_1\lambda_2=1, \quad \text{and } \Sigma\frac{1}{\lambda}=0.$$

The elimination of λ_5, λ_6 from these conditions gives the one necessary and sufficient condition that four points on the cubic should be concyclic.

Example (i). *A circle passing through the cusp cuts a semi-cubical parabola in four points; the sum of the inclinations of the radii vectores of these points to the cuspidal tangent is constant and equal to α; prove that the centre of the circle lies on the line*

$$y = x\tan\alpha.$$

Let the equation of the circle be

$$x^2+y^2+2gx+2fy = 0;$$

the equation giving the parameters of the four points of intersection is

$$a\lambda^4+a\lambda^2+2f\lambda+2g = 0.$$

Hence $\qquad \Sigma\lambda = 0, \quad \Sigma\lambda_1\lambda_2 = 1, \quad \Sigma\lambda_1\lambda_2\lambda_3 = -\dfrac{2f}{a},$

and $\qquad \lambda_1\lambda_2\lambda_3\lambda_4 = \dfrac{2g}{a}.$

But

$$\tan\alpha = \frac{\Sigma\lambda-\Sigma\lambda_1\lambda_2\lambda_3}{1-\Sigma\lambda_1\lambda_2+\lambda_1\lambda_2\lambda_3\lambda_4},$$

$$= \frac{f}{g}.$$

Hence the centre lies on the line

$$y = x\tan\alpha.$$

Example (ii). *Find the locus of the intersections of tangents to the curve* $\qquad 27ay^2 = 2(x-2a)^3$ *which are at right angles.*

The coordinates of any point on the curve satisfy the equations

$$\frac{x-2a}{3\lambda^2} = \frac{y}{\lambda^3} = \frac{a}{2} = \frac{x}{3\lambda^2+4}.$$

The equation of the tangent at the point λ is

$$2\lambda(x-2a)-4y-a\lambda^3 = 0. \qquad \ldots \qquad \ldots \qquad \text{(I)}$$

Let (x, y) be the point of intersection of the tangents, the points of contact of the three tangents from this point to the curve are given by equation (I); let the parameters be $\lambda_1, \lambda_2, \lambda_3$. Since two of the tangents are at right angles,

(1) $\lambda_1\lambda_2+4 = 0.$

Further (2) $\lambda_1+\lambda_2+\lambda_3 = 0.$

(3) $\lambda_1\lambda_2+\lambda_3(\lambda_1+\lambda_2) = -\dfrac{2(x-2a)}{a}.$

(4) $\lambda_3\lambda_1\lambda_2 = -\dfrac{4y}{a}.$

Hence $\qquad \lambda_3 = \dfrac{y}{a}.$

Therefore from (3) $\qquad +4+\dfrac{y^2}{a^2} = \dfrac{2(x-2a)}{a}.$

The required locus is consequently

$$y^2+8a^2 = 2ax.$$

Example (iii). *The tangents to the curve $x^2(x-a) = y^3$ at the points P, Q, R meet at A, and the circle PQR meets the curve again at S and at the cusp. Show that A must lie on $3x-a = 0$ and that S is a fixed point.*

The coordinates of any point on the curve are given by

$$\frac{x}{\lambda^3} = \frac{x-a}{1} = \frac{y}{\lambda^2} = \frac{a}{\lambda^3-1}.$$

The equation of the tangent at the point λ is

$$\lambda^3(x-a)-3\lambda y+2x = 0. \qquad \ldots \qquad \ldots \qquad \text{(I)}$$

Hence, if A is the point (x, y), and λ_1, λ_2, λ_3 are the points P, Q, R,

$$\lambda_1+\lambda_2+\lambda_3 = 0.$$

Again, the equation of any circle passing through the cusp $(0, 0)$ is

$$x^2+y^2+2gx+2fy = 0.$$

The intersections of this circle and the curve are given by

$$(a+2g)\lambda^4+2f\lambda^3+a\lambda^2-2g\lambda-2f = 0. \ \ldots \qquad \ldots \qquad \text{(II)}$$

If these intersections are P, Q, R, S, and if S is the point μ, then

$$\lambda_1+\lambda_2+\lambda_3+\mu = \lambda_1\lambda_2\lambda_3\mu.$$

But $$\lambda_1+\lambda_2+\lambda_3 = 0.$$

Hence $$\lambda_1\lambda_2\lambda_3 = 1.$$

Now from the equation of the tangent (I) it follows that

$$\frac{2x}{a-x} = \lambda_1\lambda_2\lambda_3 = 1.$$

Therefore the point A lies on the line

$$3x-a = 0.$$

Also, from the equation (II),

$$(\lambda_1+\lambda_2+\lambda_3)\mu+\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1 = \frac{a}{a+2g},$$

and $$\mu(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)+\lambda_1\lambda_2\lambda_3 = \frac{2g}{a+2g}.$$

Hence $$\Sigma\lambda_1\lambda_2 = \frac{a}{a+2g},$$

and $$\mu\Sigma\lambda_1\lambda_2 = \frac{2g}{a+2g}-1 = -\frac{a}{a+2g}.$$

Hence $\mu = -1$, and S is the point $\left(\frac{a}{2}, -\frac{a}{2}\right)$ which is fixed.

Example (iv). *Show that two conics can be drawn to pass through the centroid and the angular points of the triangle of reference and to touch the curve whose equation in triangular coordinates is* $y^3 = x^2 z.$

Show also that the equation of the line joining the points of contact of the cubic and conics is $3y = 2x + z.$

Let the equation of the conic be

$$lyz + mzx + nxy = 0.$$

Since it passes through the centroid,

$$l + m + n = 0.$$

Any point on the cubic is given by

$$\frac{x}{1} = \frac{y}{\lambda} = \frac{z}{\lambda^3}.$$

Hence the cubic meets the conic in points given by

$$l\lambda^3 + m\lambda^2 + n = 0.$$

Since $\qquad l + m + n = 0,$

this reduces to $\qquad l(1-\lambda^3) + m(1-\lambda^2) = 0.$

The factor $\lambda - 1$ corresponds to the centroid which lies on both curves, the other points of intersection are given by

$$l\lambda^2 + (l+m)\lambda + l + m = 0.$$

Hence, if these points are coincident,

$$(l+m)^2 = 4l(l+m),$$

or, $\qquad m = -l$ or $3l.$

The two corresponding conics are, since

$$l + m + n = 0,$$

$$z(y-x) = 0,$$

and $\qquad yz + 3zx - 4xy = 0.$

The first of these is a pair of lines.

If the conic touches the cubic at the centroid of the triangle, one root of the equation $\qquad l\lambda^2 + (l+m)\lambda + l + m = 0$

is 1,

hence $\qquad 3l + 2m = 0,$

or, $\qquad l : m : n = 2 : -3 : 1.$

Consequently the equation of the second conic touching the cubic is

$$2yz - 3xz + xy = 0.$$

The points of contact are the centroid ($\lambda = 1$) and the point $\lambda = -2$, given by the equation

$$l\lambda^2 + (l+m)\lambda + l + m = 0,$$

where $$m = 3l.$$

The chord joining the points λ_1, λ_2 on the cubic is

$$x(\lambda_1+\lambda_2)\lambda_1\lambda_2 - (\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)y + z = 0,$$

hence the equation of the line joining the points of contact is

$$2x - 3y + z = 0.$$

Example (v). *Prove that the locus of the intersection of a tangent to the curve*

$$ay^2 = x^3$$

with a perpendicular tangent to

$$ax^2 = y^3$$

is $$27(x^2+y^2)(x-y)^2 = 4a(x+y)^3.$$

The coordinates of points on these curves respectively are given by

$$(1)\quad \frac{x}{\lambda} = \frac{y}{1} = \frac{a}{\lambda^3}.$$

$$(2)\quad \frac{x}{1} = \frac{y}{\mu} = \frac{a}{\mu^3}.$$

The tangents at the points λ, μ to the two curves respectively are

$$-3x\lambda^2 + 2y\lambda^3 + a = 0,$$
$$2x\mu^3 - 3y\mu^2 + a = 0.$$

These are perpendicular if $\lambda = -\mu$.

The required locus is consequently that of the intersections of

$$2y\lambda^3 - 3x\lambda^2 + a = 0.$$

and $$2x\lambda^3 + 3y\lambda^2 - a = 0,$$

for varying values of λ.

Add, then $$\lambda = \frac{3(x-y)}{2(x+y)};$$

hence the required locus is

$$27(x^2+y^2)(x-y)^2 = 4a(x+y)^3.$$

Example (vi). *Express the equation of the tangent to*

$$y^2(2a-x) = x^3$$

in terms of the inclination θ of the radius vector to a point to the axis of x.

Show that four normals meet in a point (x, y) such that

$$\tan(\alpha+\beta+\gamma+\delta) = \frac{y}{a+x}.$$

The coordinates of any point on the cubic are given by

$$\frac{x}{\tan^2\theta} = \frac{y}{\tan^3\theta} = \frac{2a-x}{1} = \frac{2a}{\sec^2\theta}.$$

The tangent at the point θ is

$$(3\tan\theta+\tan^3\theta)x-2y-2a\tan^3\theta = 0.$$

The normal to the curve at the point θ is consequently

$$2(x-2a\tan^2\theta\cos^2\theta)+(3\tan\theta+\tan^3\theta)(y-2a\tan^3\theta\cos^2\theta) = 0.$$

This reduces at once to

$$2x+(3\tan\theta+\tan^3\theta)y-2a\tan^2\theta(2+\tan^2\theta) = 0,$$

which may also be written

$$2a\tan^4\theta-y\tan^3\theta+4a\tan^2\theta-3y\tan\theta-2x = 0.$$

Since this equation is the condition that the normal at any point θ should pass through the point (x, y), it, conversely, gives the feet of the normals which can be drawn from the point (x, y) to the curve; it is quartic in $\tan\theta$, hence four normals can be drawn.

Let $\alpha, \beta, \gamma, \delta$ be the values of θ given by this equation, then

$$\tan(\alpha+\beta+\gamma+\delta) = \frac{\Sigma\tan\alpha-\Sigma\tan\alpha\tan\beta\tan\gamma}{1-\Sigma\tan\alpha\tan\beta+\tan\alpha\tan\beta\tan\gamma\tan\delta}$$

$$= \frac{\dfrac{y}{2a}-\dfrac{3y}{2a}}{1-2-\dfrac{x}{a}} = \frac{y}{a+x}.$$

§ 53. *Nodal cubics.*

The equation of a nodal cubic can be put in the form

$$(x^2 \pm y^2)z = x^3.$$

The positive sign corresponds to cubics which have a conjugate point; these cubics are referred to as 'acnodal

cubics.' The negative sign corresponds to cubics which have a node with real tangents; these are referred to as 'crunodal cubics.'

The coordinates of any point on the nodal cubic satisfy the equations

$$\frac{x}{1 \pm \lambda^2} = \frac{y}{\lambda(1 \pm \lambda^2)} = \frac{z}{1}.$$

For acnodal cubics the conjugate point is $\pm\sqrt{-1}$; for crunodal cubics the node is ± 1, the two signs corresponding to the two values of λ for the same point on different branches of the curve.

The point $x = z = 0$ is a point of inflexion on both cubics; this is the point $\lambda = \infty$.

NOTE (1). The straight line joining the point λ on the curve to the point of inflexion $(0:1:0)$ is

$$x = (1 \pm \lambda^2)\, z.$$

This clearly also passes through the point $-\lambda$; the points $\pm\lambda$ are called corresponding points on the curve.

NOTE (2). The parameters of the points of intersection of the cubic and any straight line

$$Ax + By + Cz = 0$$

are given by the equation

$$A(1 \pm \lambda^2) + B\lambda(1 \pm \lambda^2) + C = 0.$$

If λ_1, λ_2, λ_3 be the values of λ given by this equation, since the coefficients of λ^3 and λ are numerically equal,

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \pm 1.$$

Hence,

1. This is the necessary and sufficient condition that the three points λ_1, λ_2, λ_3 should be collinear.

2. If the straight line be a tangent to the curve, two of the values of λ are equal; suppose $\lambda_2 = \lambda_3 = \lambda$, then

$$2\lambda\lambda_1 + \lambda^2 = \pm 1.$$

Hence the tangent at the point λ meets the curve again at the point

$$\frac{1}{2}\left(\pm\frac{1}{\lambda}-\lambda\right).$$

3. If the straight line meets the curve in three coincident points at λ, then

$$3\lambda^2 = \pm 1.$$

For the acnodal cubic $\lambda = \pm\sqrt{\tfrac{1}{3}}$, which consequently give two real points of inflexion other than the point $(0:1:0)$. These two points of inflexion are corresponding points, and hence the line joining them passes through the third point of inflexion.

For the crunodal cubic $\lambda = \pm\sqrt{-\tfrac{1}{3}}$, which values correspond to two imaginary points of inflexion. The only real point of inflexion on a crunodal cubic is consequently the point

$$x = z = 0.$$

NOTE (3). The equation of the chord joining any two points λ_1, λ_2 on the cubic is

$$x\left[1\pm(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)\right]\mp(\lambda_1+\lambda_2)\,y-(1\pm\lambda_1)\,(1\pm\lambda_2^2)\,z = 0.$$

NOTE (4). The equation of the tangent at the point λ is, consequently,

$$x\,(1\pm 3\lambda^2)\mp 2\lambda y-(1\pm\lambda^2)^2 z = 0,$$

or,
$$z\lambda^4\pm(2z-3x)\,\lambda^2\pm 2y\lambda+z-x = 0.$$

This equation is quartic in the variable λ, hence four tangents, real or imaginary, can be drawn from any point $(x:y:z)$ to a nodal cubic; the parameters of the points of contact are given by this equation.

If λ_1, λ_2, λ_3, λ_4 are the points of contact of the four tangents from the point $(x:y:z)$ to a nodal cubic, since the coefficient of λ^3 is zero,

$$\Sigma\lambda = 0. \qquad \ldots \qquad \ldots \qquad \text{(I)}$$

Also
$$\pm\Sigma\lambda_1\lambda_2 = 2-\frac{3x}{z},$$

and
$$\lambda_1\lambda_2\lambda_3\lambda_4 = 1-\frac{x}{z};$$

therefore
$$\pm\Sigma\lambda_1\lambda_2-3\lambda_1\lambda_2\lambda_3\lambda_4+1=0. \qquad \ldots \qquad \text{(II)}$$

These relations (I) and (II) are independent of the coordinates x, y and z; and therefore they are the necessary and sufficient conditions that the tangents at the four points $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ should be concurrent.

If λ_4 be eliminated from these two conditions, the eliminant gives the single condition that the tangents at three points should be concurrent.

Example (i). *In the curve known as the Witch of Agnesi, $xy^2 = a^2(a-x)$, show that the tangent at the point (x, y) meets the curve again at the point*
$$X = \frac{4x^2y^2}{a^3}; \quad Y = a^2 \cdot \frac{2x-a}{2xy}.$$

The equation of the curve can be written
$$x(y^2+a^2) = a^3.$$

The coordinates of any point on the curve satisfy the equations
$$\frac{x}{1} = \frac{y}{\lambda(1+\lambda^2)} = \frac{a}{1+\lambda^2} = \frac{2x-a}{1-\lambda^2}.$$

If the tangent at the point λ meets the curve at the point λ_1, then
$$2\lambda\lambda_1+\lambda^2 = 1,$$

or,
$$\lambda_1 = \frac{1-\lambda^2}{2\lambda}.$$

Hence, if λ is the point (x, y) and λ_1 the point (X, Y), since
$$1+\lambda_1^2 = 1+\frac{(1-\lambda^2)^2}{4\lambda^2} = \frac{(1+\lambda^2)^2}{4\lambda^2},$$

therefore
$$X = \frac{a}{1+\lambda_1^2} = \frac{4a\lambda^2}{(1+\lambda^2)^2} = \frac{4x^2y^2}{a^3},$$

and
$$Y = a\lambda_1 = a\cdot\frac{1-\lambda^2}{2\lambda} = a^2\cdot\frac{2x-a}{2xy}.$$

Example (ii). *Two points on the curve,* $a(x^2-y^2)=x^3$, *subtend a right angle at the origin; find the locus of the intersection of the tangents at these points.*

The coordinates of any point on the curve satisfy the equations

$$\frac{x}{1-\lambda^2}=\frac{y}{\lambda(1-\lambda^2)}=\frac{a}{1}.$$

If λ, λ' are two of the points in question

$$\lambda\lambda'=-1.$$

The tangents at the points λ, $-\frac{1}{\lambda}$ are

$$x(1-3\lambda^2)+2\lambda y-(1-\lambda^2)^2a=0, \quad \ldots \quad \ldots \quad \text{(I)}$$

$$x\left(1-\frac{3}{\lambda^2}\right)-\frac{2y}{\lambda}-\left(1-\frac{1}{\lambda^2}\right)^2a=0,$$

or,
$$x(\lambda^4-3\lambda^2)-2\lambda^3y-(1-\lambda^2)^2a=0. \quad \ldots \quad \ldots \quad \text{(II)}$$

The required locus is obtained by eliminating λ from these equations. Subtract (II) from (I), and divide by $(1+\lambda^2)$. It then follows that

$$x(1-\lambda^2)+2\lambda y=0.$$

Hence, substituting in equation (I),

$$x(1-3\lambda^2)+2\lambda y-\frac{4\lambda^2y^2a}{x^2}=0,$$

or,
$$\lambda^2\left(3x+\frac{4y^2a}{x^2}\right)-2\lambda y-x=0$$

But
$$\lambda^2x-2\lambda y-x=0.$$

Hence
$$2x+\frac{4y^2a}{x^2}=0,$$

or,
$$x^3+2ay^2=0,$$

which is the equation of the required locus.

Example (iii). *Find the tangents to the curve,* $a(x^2+y^2)=x^3$, *which are inclined at an angle of* 60° *to the axis of* x, *and show that their points of contact are points of inflexion.*

The coordinates of any point on the curve satisfy the equations

$$\frac{x}{1+\lambda^2}=\frac{y}{\lambda(1+\lambda^2)}=\frac{a}{1}.$$

The equation of the tangent at the point λ is

$$x(1+3\lambda^2)-2\lambda y-a(1+\lambda^2)^2=0.$$

The inclination of this tangent to the axis of x is

$$\tan^{-1}\frac{1+3\lambda^2}{2\lambda}.$$

Hence the points of contact of the tangents in the equation are given by

$$\frac{1+3\lambda^2}{2\lambda} = \pm\sqrt{3},$$

i.e.

$$\lambda = \pm\frac{1}{\sqrt{3}}.$$

These points are points of inflexion (§ 10, Note a); the equations of the corresponding tangents are

$$9x \mp 3\sqrt{3}y - 8a = 0.$$

Example (iv). *A circle passes through the origin and touches the curve* $x^3 = y^3 - y^2$; *find the locus of its centre.*

Let the equation of the circle be

$$x^2+y^2-2gx-2fy = 0.$$

The parameters of its points of intersection with the given cubic are obtained by substituting $x = 1-\lambda^2$; $y = \lambda(1-\lambda^2)$ in this equation.

Hence $$\lambda^4+2f\lambda+2g-1 = 0.$$

Since the circle touches the curve, two of the roots of this equation are equal. Let the roots be $\lambda, \lambda, \lambda_1, \lambda_2$.

Hence $$\lambda_1+\lambda_2 = -2\lambda.$$

$$\lambda^2+\lambda_1\lambda_2-2\lambda(\lambda_1+\lambda_2) = 0,$$

i.e. $$\lambda_1\lambda_2 = 3\lambda^2.$$

But $$-2f = \lambda^2(\lambda_1+\lambda_2)+2\lambda\lambda_1\lambda_2,$$

$$= 4\lambda^3,$$

or, $$f = -2\lambda^3,$$

and $$2g-1 = \lambda^2\lambda_1\lambda_2 = 3\lambda^4.$$

Therefore $$16(2g-1)^3 = 27f^4.$$

The required locus is accordingly

$$16(2x-1)^3 = 27y^4.$$

Example (v). *An harmonic pencil, whose vertex is at the node, meets a nodal cubic* $a(x^2-y^2) = x^3$ *at four points the tangents at which are concurrent; prove that the locus of the intersections of these tangents is another cubic.*

The coordinates of any point on the cubic are given by

$$\frac{x}{1-\lambda^2} = \frac{y}{\lambda(1-\lambda^2)} = \frac{a}{1}.$$

The parameters of the points of contact of tangents from the point (x, y) to the curve are given by the equation

$$a\lambda^4 + \lambda^2(3x-2a) - 2\lambda y + a - x = 0.$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the points of contact.

Since the pencil $y-\lambda_1 x = 0$; $y-\lambda_2 x = 0$; $y-\lambda_3 x = 0$; $y-\lambda_4 x = 0$ is harmonic,
$$(\lambda_1+\lambda_2)(\lambda_3+\lambda_4) = 2\lambda_1\lambda_2 + 2\lambda_3\lambda_4.$$

But
$$\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 0.$$

Hence
$$\lambda_1\lambda_2+\lambda_3\lambda_4 = -\tfrac{1}{2}(\lambda_1+\lambda_2)^2,$$

and
$$3(\lambda_1\lambda_2+\lambda_3\lambda_4) = \Sigma\lambda_1\lambda_2 = \frac{3x-2a}{a}.$$

Also
$$\lambda_1\lambda_2(\lambda_3+\lambda_4)+\lambda_3\lambda_4(\lambda_1+\lambda_2) = \frac{2y}{a},$$

i.e.
$$(\lambda_1+\lambda_2)(\lambda_1\lambda_2-\lambda_3\lambda_4) = -\frac{2y}{a},$$

or,
$$(\lambda_1\lambda_2-\lambda_3\lambda_4)^2 = \frac{6ay^2}{a(2a-3x)}.$$

Hence
$$\left(\frac{3x-2a}{3a}\right)^2 + \frac{6ay^2}{a(3x-2a)} = 4\lambda_1\lambda_2\lambda_3\lambda_4 = \frac{4(a-x)}{a},$$

which reduces to

$$(3x-2a)^3 + 54ay^2 + 36a(x-a)(3x-2a) = 0.$$

§ **54.** In many other special cases a convenient expression for the coordinates of any point on a curve in terms of a single parameter can be found. Some more general illustrative examples follow; the reader will find numerous easy applications of the methods illustrated in this chapter among the exercises at the end of the book. The construction of problems is one of the most useful exercises the student can undertake, and in the case of unicursal cubic curves there is a wide scope for ingenuity.

Example (i). *The nine points $ABCDEFGHI$ are so situated that ABC lie on a straight line, and so do DEF, GHI, ADG, BEH, CFI. Through these nine points a cubic curve is drawn which has an asymptote parallel to a straight line which meets the six straight lines in L, M, N, X, Y, Z and the cubic again in P and another point. Show that*

$$PL \,.\, PM \,.\, PN = PX \,.\, PY \,.\, PZ.$$

Let the equations of the six straight lines be

$$a_1 x + b_1 y + 1 = 0, \qquad A_1 x + B_1 y + 1 = 0,$$
$$a_2 x + b_2 y + 1 = 0, \qquad A_2 x + B_2 y + 1 = 0,$$
$$a_3 x + b_3 y + 1 = 0, \qquad A_3 x + B_3 y + 1 = 0.$$

The equation of the cubic is

$$(a_1 x + b_1 y + 1)(a_2 x + b_2 y + 1)(a_3 x + b_3 y + 1)$$
$$= p(A_1 x + B_1 y + 1)(A_2 x + B_2 y + 1)(A_3 x + B_3 y + 1).$$

Suppose that the straight line $y = 0$ is an asymptote of this curve, since it must meet the curve in two points at infinity, the coefficients of x^3 and x^2 must be zero in the equation

$$(a_1 x + 1)(a_2 x + 1)(a_3 x + 1) = p(A_1 x + 1)(A_2 x + 1)(A_3 x + 1).$$

Hence $$a_1 a_2 a_3 = p A_1 A_2 A_3.$$

Let the straight line LMN be $y = k$, and suppose this line meets the cubic in the point (h, k).

The point L is $$\left(-\frac{b_1 k + 1}{a_1},\ k\right).$$

Hence $PL = \dfrac{a_1 h + b_1 k + 1}{a_1}$. Similar expressions give the lengths PM, PN, PX, PY, PZ.

Since the point (h, k) is on the cubic

$$\frac{(a_1 h + b_1 k + 1)(a_2 h + b_2 k + 1)(a_3 h + b_3 k + 1)}{a_1 a_2 a_3}$$
$$= \frac{(A_1 h + B_1 k + 1)(A_2 h + B_2 k + 1)(A_3 h + B_3 k + 1)}{A_1 A_2 A_3},$$

which is equivalent to the statement

$$PL \,.\, PM \,.\, PN = PX \,.\, PY \,.\, PZ.$$

Example (ii). *The cubic $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0$ meets the side BC of the triangle of reference ABC in D. If the tangent at some other point P passes through D, show that AP passes through the conjugate point of the curve, and that AB, AC, AD, AP form an harmonic pencil.*

The coordinates of any point on the curve satisfy the implicit equations
$$\frac{x}{\lambda^3}=\frac{y}{(1-\lambda)^3}=\frac{z}{-1}.$$
The cubic cuts the straight line BC ($x=0$) at the point $\lambda=0$, hence D is the point 0 and its coordinates are $(0:1:-1)$. The equation of AD is accordingly $y+z=0$. The intersections of the cubic and any straight line $Ax+By+z=0$ are given by the equation
$$A\lambda^3+B(1-\lambda)^3-1=0.$$
Hence, if the parameters of these points of intersection are λ_1, λ_2, λ_3, since the coefficients of λ^2 and λ are $-3B$ and $3B$, it follows that
$$\Sigma\lambda=\Sigma\lambda_1\lambda_2;$$
if then the straight line is a tangent to the curve at the point λ and meets the curve again at the point λ_1,
$$2\lambda+\lambda_1=\lambda^2+2\lambda\lambda_1.$$
Consequently if λ_1 is the point D, i.e. $\lambda_1=0$, P is the point $\lambda=2$; the coordinates of the point P are $(8:-1:-1)$, and the equation of AP is $y-z=0$.

The equations of the curve can be written
$$\frac{x-z}{1+\lambda^3}=\frac{y-z}{1+(1-\lambda)^3}=\frac{z}{-1}.$$
But $1+\lambda^3$ and $1+(1-\lambda)^3$ have the common quadratic factor $1-\lambda+\lambda^2$, whose linear factors are imaginary, hence the point
$$x-z=y-z=0$$
is a conjugate point; this point clearly lies on the straight line AP $(y-z=0)$.

Since the equations of the lines AB, AC, AD, AP are
$$z=0;\quad y=0;\quad y+z=0;\quad \text{and}\quad y-z=0,$$
they form an harmonic pencil.

Example (iii). *A tangent at the point P of the curve $x^3+y^3=2a^3$ meets the curve again in Q; A, B, C are the points where it meets $x^3+y^3+6axy=2a^3$; show that*
$$\frac{3}{PQ}=\frac{1}{PA}+\frac{1}{PB}+\frac{1}{PC}.$$

Let P be the point (α, β), and $PQABC$ the straight line

$$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta} = r.$$

Since PQ is the tangent at P, the roots of the equation in r,

$$(r\cos\theta+\alpha)^3+(r\sin\theta+\beta)^3 = 2a^3, \quad \ldots \quad \ldots \quad \text{(I)}$$

are $$PQ,\ 0,\ 0.$$

The roots of the equation

$$(r\cos\theta+\alpha)^3+(r\sin\theta+\beta)^3+6a(r\cos\theta+\alpha)(r\sin\theta+\beta) = 2a^3 \quad \text{(II)}$$

are evidently $$PA,\ PB, \text{ and } PC.$$

From (I), it follows that

$$\alpha^2\cos\theta+\beta^2\sin\theta = 0;$$

therefore $$\tan\theta = -\frac{\alpha^2}{\beta^2}.$$

From equation (II), remembering that $\alpha^3+\beta^3 = 2a^3$,

$$\frac{1}{PA}+\frac{1}{PB}+\frac{1}{PC} = \frac{\beta\cos\theta+\alpha\sin\theta}{\alpha\beta} = \frac{\beta^3-\alpha^3}{\alpha\beta^3}\cos\theta.$$

But $$\frac{3}{PQ} = \frac{\cos^3\theta+\sin^3\theta}{\alpha\cos^2\theta+\beta\sin^2\theta} = \frac{1+\tan^3\theta}{\alpha+\beta\tan^2\theta}\cos\theta.$$

$$= \frac{\beta^6-\alpha^6}{\alpha\beta^3(\beta^3+\alpha^3)}\cos\theta = \frac{\beta^3-\alpha^3}{\alpha\beta^3}\cos\theta.$$

Therefore $$\frac{3}{PQ} = \frac{1}{PA}+\frac{1}{PB}+\frac{1}{PC}.$$

Example (iv). *If two points on the hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}} = c^{\frac{2}{3}}$ are such that the tangent at P is the normal at Q, find the length of PQ.*

The coordinates of any real point on the curve are each numerically less than c, hence any point on the curve can be represented by $(c\cos^3\theta, c\sin^3\theta)$.

Let $Ax+By+c = 0$ be the equation of the chord joining the points θ and ϕ.

Then $$A\cos^3\theta+B\sin^3\theta+1 = 0$$
and $$A\cos^3\phi+B\sin^3\phi+1 = 0.$$

Hence, by cross-multiplication,

$$\frac{A}{\sin^3\theta-\sin^3\phi} = \frac{B}{\cos^3\theta-\cos^3\phi} = \frac{1}{\sin^3\phi\cos^3\theta-\sin^3\theta\cos^3\phi}.$$

Divide each denominator by $\sin\frac{\theta-\phi}{2}$ (vide § 49) and put $\theta=\phi$ in the result. It then follows that

$$\frac{A}{\sin\theta}=\frac{B}{\cos\theta}=\frac{1}{-\sin\theta\cos\theta}.$$

Hence the equation of the tangent at the point θ is

$$x\sin\theta+y\cos\theta=c\cos\theta\sin\theta.$$

Hence the equation of the normal at ϕ is

$$x\cos\phi-y\sin\phi=c\cos 2\phi.$$

If these equations are identical

$$\frac{\sin\theta}{\cos\phi}=\frac{\cos\theta}{-\sin\phi}=\frac{\sin 2\theta}{2\cos 2\phi}.$$

Therefore $\qquad \tan\theta=-\cot\phi,$

or, $\qquad \theta=\frac{\pi}{2}+\phi.$

Hence $\qquad \tan 2\theta=-2$ and $\sin^2 2\theta=\frac{4}{5}.$

Now
$$\begin{aligned}\frac{PQ^2}{c^2}&=(\cos^3\theta+\sin^3\theta)^2+(\cos^3\theta-\sin^3\theta)^2,\\&=2(\cos^6\theta+\sin^6\theta),\\&=2(1-\tfrac{3}{4}\sin^2 2\theta),\\&=\tfrac{4}{5}.\end{aligned}$$

Hence $\qquad PQ=\frac{2c}{\sqrt{5}}.$

Example (v). *A chord of the cubic $a(x^2+y^2)=xy^2$ subtends a right angle at the conjugate point; show that this chord touches a fixed parabola.*

The coordinates of any point on the cubic satisfy the equations

$$\frac{x}{1+\lambda^2}=\frac{y}{\lambda(1+\lambda^2)}=\frac{a}{\lambda^2}.$$

The equation of the chord joining two points λ_1, λ_2 is

$$\lambda_1\lambda_2(1-\lambda_1\lambda_2)x-(\lambda_1+\lambda_2)y+a(1+\lambda_1^2)(1+\lambda_2^2)=0.$$

If the chord subtends a right angle at the origin

$$\lambda_1\lambda_2=-1.$$

The equation of the chord then becomes

$$2x+(\lambda_1+\lambda_2)y-\{4+(\lambda_1+\lambda_2)^2\}a=0.$$

Hence it touches the parabola

$$y^2+8a(x-2a)=0.$$

Example (vi). *The circle of curvature at a point P of the curve $axy = x^3+y^3$ passes through the origin and meets the curve again at Q; show that PQ touches the curve $4xy = a^2$.*

The coordinates of any point on the cubic satisfy the equations

$$\frac{x}{\lambda} = \frac{y}{\lambda^2} = \frac{a}{1+\lambda^3}.$$

The equation of any circle through the origin is

$$x^2+y^2+2gx+2fy = 0.$$

The points of intersection of this circle and the curve are given by

$$a\lambda+a\lambda^3+2g(1+\lambda^3)+2f\lambda(1+\lambda^3) = 0.$$

If these points are $P(\lambda)$ and $Q(\lambda_1)$ the roots of this equation are $\lambda, \lambda, \lambda, \lambda_1$.

Since the coefficient of λ^2 is zero

$$3\lambda^2+3\lambda\lambda_1 = 0,$$

or,

$$\lambda+\lambda_1 = 0.$$

Hence, if P is the point λ, Q is the point $-\lambda$, and the equation of the chord PQ,

$$\lambda^4x-\lambda^2a+y = 0.$$

This chord consequently touches the rectangular hyperbola

$$4xy = a^2.$$

EXAMPLES FOR EXERCISE

1. Any straight line through a fixed point $O\,(h, k)$ meets the straight lines $ax^2+2hxy+by^2=0$ at the points P, Q. A third point R is taken on this line so that

$$\text{(1)}\quad OR^{-1} = OP^{-1}+OQ^{-1}.$$
$$\text{(2)}\quad OR = OP+OQ.$$
$$\text{(3)}\quad OR^2 = OP\,.\,OQ.$$

Find the locus of R in each case.

2. Show that the conditions, that the straight lines $ax^2+2hxy+by^2=0$ should form an equilateral triangle with $x\cos\alpha+y\sin\alpha = p$, are

$$\frac{a}{1-2\cos^2\alpha} = \frac{h}{2\sin^2\alpha} = \frac{b}{1+\cos 2\alpha}.$$

3. The connector of any point $P\,(X, Y)$ with the origin meets the straight line $ax+by+1 = 0$ in Q; show that

$$PQ : OQ = aX+bY+1.$$

4. If $\tan\alpha\tan\beta$ is constant, show that the locus of the intersections of the straight lines

$$x\sec 2\alpha - y\operatorname{cosec} 2\alpha = 1 = x\sec 2\beta - y\operatorname{cosec} 2\beta$$

is an ellipse.

5. Find the area of the triangle formed by the right lines $lx+my+1=0$, $ax^2+2hxy+by^2 = 0$; and if the sides of the triangle which meet at the origin are equal in length, prove that

$$\frac{l^2-m^2}{a-b} = \frac{lm}{h}.$$

6. Find the area of the triangle formed by the three points where the circle $x^2+y^2 = 2ax+2by$ is cut by the pair of straight lines

$$lx^2+2mxy+ny^2 = 0.$$

7. Any straight line through the origin meets the curve $x^3+y^3 = a^3$ in the points P, Q, and R. Find the locus of the centre of mean position of these points.

8. Find the condition that the three straight lines

$$x\cos 3\alpha+y-a\cos\alpha = 0,$$
$$x\cos 3\beta+y-a\cos\beta = 0,$$
$$x\cos 3\gamma+y-a\cos\gamma = 0$$

should be concurrent.

9. A straight line through the origin meets the line $ax+by+c=0$ at P. Find the locus of a point Q on this straight line such that PQ is a constant length.

10. Find the equation of the inverse of the ellipse $x^2+2y^2=2$ with respect to a circle whose centre is the point (1, 0) and whose radius is unity.

11. Find the length of the intercept made on the line $y=x\tan\theta+c$ by the lines $ax^2+2hxy+by^2=0$. Also when the axes are oblique.

12. Prove that the equation

$$5(x^2+y^2)^2(a-x)-20(x^2+y^2)(a^3-x^3)+16(a^5-x^5)=0$$

represents the sides of a regular pentagon.

13. Two lines AP, BP rotate with equal and opposite angular velocities about fixed points A and B. Find the locus of P.

14. Find the condition that the three straight lines

$$x\cos\alpha+y\sin\alpha=a\cos3\alpha,$$
$$x\cos\beta+y\sin\beta=a\cos3\beta,$$
$$x\cos\gamma+y\sin\gamma=a\cos3\gamma,$$

should be concurrent.

15. Show that the coordinates of the orthocentre of the triangle formed by the straight lines $ax^2+2hxy+by^2=0$ and $lx+my=1$ are given by

$$\frac{x}{l}=\frac{y}{m}=\frac{a+b}{am^2-2hlm+bl^2}.$$

16. Show that the equation of the circle whose diameter is the intercept made on the line $lx+my=1$ by the lines $ax^2+2hxy+by^2=0$ is

$$(x^2+y^2)(am^2-2hlm+bl^2)+2x(hm-bl)+2y(hl-am)+a+b=0.$$

17. Show that the inverse of a conic with respect to any point is a quartic. What does it become when the conic is a circle?

18. If the four straight lines obtained by substituting α, β, γ, δ for θ in $x\sin(\theta+a)=y\sin2\theta+b$ are concurrent, the sum of the angles α, β, γ, δ is $(2n+1)\pi$.

19. Find the condition that the three straight lines

$$x\cos\alpha+y\sin\alpha+a\cos2\alpha=0,$$
$$x\cos\beta+y\sin\beta+a\cos2\beta=0,$$
$$x\cos\gamma+y\sin\gamma+a\cos2\gamma=0,$$

should be concurrent, and find the equation of a fourth line of similar form through their point of intersection.

20. If the one vertex of an equilateral triangle is fixed and another lies on a fixed line, find the locus of the centroid.

21. Find the locus of the middle point of the intercept made by the coordinate axes on a straight line passing through a fixed point.

22. One of the lines $ax^2+bxy+cy^2=0$ coincides with one of the lines $a'x^2+b'xy+c'y^2=0$; if ϕ be the angle between the other two, prove that $(ac'-a'c)^2\cot\phi = aa'(bc'-b'c)+cc'(ab'-a'b)$.

23. A series of parallel chords are drawn to a curve of the nth degree, and on each chord a point is taken which is the centre of mean position of the n points of intersection of the chord and curve. Show that the locus of this point is a line.

24. A rectangular hyperbola has a given focus and passes through a given point. Prove that the loci of its centre and the feet of its directrixes are the inverses of rectangular hyperbolas with respect to the common focus.

25. The equation of the bisectors of the straight lines

$$ax^2+by^2+2hxy+2gx+2fy+c=0 \text{ (where } \Delta=0, \S\S\ 6,\ 13)$$

is given by

$$\begin{vmatrix} X^2 & XY & Y^2 \\ a & h & b \\ 1 & \cos\omega & 1 \end{vmatrix} = 0,$$

where $X=ax+hy+g$, $Y=hx+by+f$, and ω is the angle between the coordinate axes.

26. The coefficients of the general equation of a conic are connected by the relation $af^2-2fgh+bg^2=0$; prove that one of the asymptotes passes through the origin.

27. Show that the equation of the directrix of the parabola

$$ax^2+2gx+2fy+c=0 \text{ is } 2afy+ac-g^2=0.$$

28. Through a fixed point P is drawn a chord QQ' of a given conic, and on it are taken points R_1, R_2, R_3 such that PR_1, PR_2, PR_3 are respectively the arithmetic, geometric, and harmonic means between PQ and PQ'. Prove that the loci of R_1 and R_2 are conics similar to the given conic, and that the locus of R_3 is a straight line.

29. Tangents are drawn to the system of conics $kx^2+y^2=2ax$, where k is arbitrary, in such a way that the intercept on each between the point of contact and the axis of y is constant. Prove that the locus of the middle point of this intercept is a circle concentric with the circle which belongs to the system of conics.

30. If $ax^2+2hxy+by^2=c$ represent an ellipse, prove that the minor axis will lie in the first and third quadrants if h and a are of the same sign.

31. Three straight lines through a point O meet a conic in the pairs of points AA', BB', CC'; prove that

$$\text{area } ABC : \text{area } A'B'C' :: OA \,.\, OB \,.\, OC : OA' \,.\, OB' \,.\, OC'.$$

32. Prove that in general two parabolas can be drawn through the points of intersection of two curves represented by general equations of the second degree, and that if the axes of these parabolas are at right angles $\quad h : h' :: a-b : a'-b'.$

33. Investigate the condition that from the point (a, b) on the circle $x(x-a)+y(y-b)=0$ it may be possible to draw two chords, each bisected by the axis of x, and show that the angle between these is

$$\frac{\tan^{-1}\sqrt{(a-8b^2)}}{3b}.$$

34. If the general equation of the second degree represents straight lines, the equation of the bisectors of the angle between them is

$$(ab-h^2)\left\{h(x^2-y^2)-(a-b)xy+2fx-2gy\right\}$$
$$+(a+b)\left\{x(gh-af)-y(fh-bg)\right\}-h(f^2-g^2)-(a-b)fg=0.$$

35. A point moves so that the sum of the squares of its distances from two given sides of an equilateral triangle is constant and equal to $2c^2$. Show that the locus is an ellipse and find the position of its foci.

36. Show that tangents to a given parabola which are inclined to each other at an angle of 45° intersect on a rectangular hyperbola.

37. Find the locus of the middle points of focal chords of a parabola.

38. Show that the locus of the point of intersection of equal chords of a parabola drawn in given fixed directions is a straight line.

39. Find the envelope of a circle whose diameter is a chord of the parabola $y^2=4ax$ passing through a fixed point on the axis of x, and show that for one position of the point the envelope reduces to a circle and a straight line.

40. The radius of curvature at any point of a parabola is double the third proportional to the perpendicular from the focus on the tangent and the focal radius vector to the point.

41. Find the equation of the osculating circle at any point of a parabola in terms of the parameter of the point, and show that the distance of the centre of the circle from the directrix is three times that of the point.

42. Find the locus of the intersection of normals to a parabola which are inclined at 45° to each other.

43. Show that the envelope of chords of a parabola which subtend an angle of 45° at the vertex is the ellipse

$$x^2+8y^2-24ax+16a^2 = 0.$$

44. Circles are described on any two focal chords of a parabola as diameters. Show that their common chord passes through the vertex.

45. Prove that the chords which pass through a point (x', y') within a parabola $y^2 = 4ax$, and are divided by it in the ratio $k:l$, have for their equation

$$4\left\{y'(y-y')-2a(x-x')\right\}^2 lk+(y-y')^2(y'^2-4ax')(k-l)^2 = 0.$$

46. PG, the normal at P to a parabola, cuts the axis at G, and is produced to Q so that $GQ = \frac{1}{2}PG$; show that the other normals passing through Q intersect at right angles.

47. P and Q are two points on a parabola the tangents at which meet in T and the normals in N. Prove that the projection of TN on the axis is equal to the sum of the distances of P and Q from the directrix.

48. If the normals at the points A, B, C on a parabola meet on the directrix at a point O, show that the points O, C, and the intersections of the tangents at A and B are collinear.

49. Find the locus of the points of trisection of parallel chords of a parabola.

50. The normals at three points on a parabola meet in a point. Show that the centres of curvature at the other extremities of the focal chords through the three points are collinear.

51. A circle of constant radius passes through the vertex of a parabola; show that the normals at the three other intersections of the circle and parabola meet in a point, and find the locus of this point.

52. Find the lengths of the normals drawn to a parabola from a point on the axis distant $8a$ from the focus.

53. Chords of the parabola $y^2 = 4ax$ pass through the foot of the directrix; show that the normals at their extremities meet on the parabola
$$y^2 = a(x-a).$$

54. If the normals at three points P, Q, R on the parabola $y^2-4ax = 0$ meet at a point whose abscissa is x, the centroid of the triangle PQR is on the axis at a distance $\frac{2}{3}(x-2a)$ from the vertex.

55. A conic has 4-point contact with the parabola $y^2 = 4ax$ and the radius of its director circle is constant and equal to c. Prove that the locus of its centre is the curve

$$(y^2 - 4ax)(y^2 + 4ax + 8a^2) + 16a^2c^2 = 0.$$

56. Show that the length of the normal other than the radius of curvature which can be drawn from the centre of curvature of a point on a parabola is

$$a\left\{3 - \left(\frac{\rho}{2a}\right)^{\frac{2}{3}}\right\}\left\{4\left(\frac{\rho}{2a}\right)^{\frac{2}{3}} - 3\right\}^{\frac{1}{2}},$$

where ρ is the radius of curvature.

57. From a point P three normals are drawn to a parabola, and a circle is drawn through the second points in which they meet the parabola. Show that if the centre of this circle lies on the axis, the locus of P is a parabola.

58. From the point (h, k) two tangents are drawn to the parabola $y^2 = 4ax$; show that the square of the area of the triangle formed by these two tangents and their chord of contact is

$$\frac{(k^2 - 4ah)^3}{a^2}.$$

59. The normal at a point of a parabola makes with the axis an angle θ. Prove that the normal at its other extremity makes an angle θ' with the axis such that

$$\tan\theta' + \tan\theta + 2\cot\theta = 0.$$

60. Find the locus of the centre of a circle circumscribing the triangle formed by the tangents from any point on a line $x = c$ to a parabola and their chord of contact.

61. Show that the four points $(0 \cdot 25, 1)$, $(2 \cdot 25, 3)$, $(1 \cdot 69, -2 \cdot 6)$, $(0 \cdot 49, -1 \cdot 4)$, lie on a circle, and find its equation.

62. A rectangular hyperbola has double contact with a parabola. Prove that the distance between the poles of its asymptotes with regard to the parabola is equal to the focal chord of the parabola parallel to the chord of contact.

63. The envelope of chords of a parabola the tangents at the ends of which include a constant angle is an ellipse.

64. An ellipse of constant eccentricity e osculates a parabola $y^2 = 4ax$. Prove that the locus of its centre is given by the equation

$$(1 - e^2)(y^2 + 4ax + 8a^2)^2 = 8a^2(2 - e^2)^2(4ax - y^2).$$

65. From any point T tangents TP, TQ are drawn to a parabola. If the circle TPQ meets in O the diameter of the parabola through T, prove that
$$OP \,.\, SP = OQ \,.\, SQ.$$

66. A triangle is self-conjugate with respect to a parabola. Prove that the perpendiculars to its sides at their points of intersection with the axis of the parabola meet in a point.

67. Find the locus of the centroid of a triangle whose sides touch a parabola, the area of whose circumscribing circle is constant and whose orthocentre is a fixed point.

68. A circle is drawn through the vertex of a parabola, meeting it in three other points, and the product of the radii of curvature at the four points is l^4, where l is the semi-latus rectum. Show that the centre of the circle lies on an ellipse passing through the focus.

69. Show that the points of intersection of the parabola $y^2 = 4ax$ with the cubic
$$2(4x^3-7xy^2-3y^3)+a(20y^2+3xy-14x^2)+3a^2(6x+7y)-36a^3 = 0$$
are such that the normals to the parabola at these points all pass through one of two points.

70. A parabola is drawn touching the sides AB, BC, CD, DA of the cyclic quadrilateral $ABCD$. Show that its directrix passes through the intersection of AC and BD.

71. The tangents at the points Q and R on the parabola $y^2 = 4ax$ intersect at P; the perpendicular from P on QR meets the axis in G. Prove that the radius of the circle PQR is $\dfrac{SP \,.\, PG}{2a}$.

72. An ellipse is drawn having for principal axes the tangents to a parabola from a point on the directrix, and touching the parabola at its vertex. Prove that the semi-latus rectum of the parabola is a mean proportional to the axes of the ellipse.

73. PQ is the common chord of a parabola $y^2 = 4ax$ and its osculating circle. Show that the locus of the intersection of PQ with the perpendicular drawn on it from the vertex is the cissoid $y^2(3a-x) = x^3$.

74. The envelope of the polar of the focus of the parabola $y^2 = 4ax$ with respect to any rectangular hyperbola which has 4-point contact with the parabola is $y^2 = 4a(3x+2a)$.

75. Tangents OP, OQ are drawn to the parabola $y^2 = 4ax$ from a point O lying on the straight line $x = -3a$; show that the envelope of the circle OPQ is $y^2(4a+x) = x(3a+x)(5a-x)$.

76. From a point P on a parabola two normals other than the normal at P are drawn to the curve; find the envelope of the chord joining their extremities and the locus of its middle point.

77. Through a fixed point O any straight line is drawn to cut a parabola in P, Q. Prove that (i) $OP.OQ$ varies as the length of the focal chord parallel to OPQ; (ii) the locus of the middle point of PQ is a parabola with vertex O and latus rectum one-half that of the given parabola.

78. The normals at the extremities of a chord which passes through the point $(-2a, 0)$ meet on the curve $y^2 = 4ax$, and contain the same angle as the lines joining the origin to the points.

79. If normals are drawn to a parabola from any point on a straight line parallel to the axis, show that the triangle formed by joining the extremities of the normals circumscribes a fixed parabola.

80. On any chord LM of a parabola as diameter a circle is described cutting the parabola again at the points N and R. If NR, LM meet the axis in P and Q show that PQ is of constant length.

81. A chord of a parabola is drawn in a fixed direction, and on it as diameter a circle is described. Prove that the polar of the vertex with regard to this circle envelopes a fixed parabola.

82. Any point P being taken on a parabola a series of points, $P_1, P_2, P_3 \ldots, P_{r-1}, P_r \ldots$, on the parabola is constructed so that the normal to P_1 passes through C, the centre of curvature at P, the normal at P_2 passes through C_1, the centre of curvature at P_1, and the normal at P_r passes through C_{r-1}, the centre of curvature at P_{r-1}. Prove that, as P moves, the envelope of the straight line $P_m P_n$ is a parabola, and the envelope of the straight line $C_m C_n$ is the evolute of a parabola, m and n being constant.

83. Show that four of the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ also touch the ellipse $(a^2 + k\alpha^2)^{-1} x^2 + (b^2 + k\beta^2)^{-1} y^2 = (l + k)^{-1}$ whatever may be the value of k.

84. Find the locus of the intersections of tangents to an ellipse which meet at a given angle.

85. Show that the locus of the middle points of chords of an ellipse, the tangents at the ends of which meet on the circle $x^2 + y^2 = a^2$, is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2 + y^2}{a^2}.$$

86. The circle of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at P meets the ellipse again at Q, and the normals at P and Q meet at G; GR, GR' being the other normals drawn to the ellipse from G, show that the tangents at R, R' intersect on the curve

$$x^2y^2(b^2x^2+a^2y^2)=(b^2x^2-a^2y^2)^2.$$

87. The normal at P to an ellipse meets the curve again in Q; show that the locus of the middle point of PQ is

$$(b^2x^2+a^2y^2)^2(b^6x^2+a^6y^2)=a^4b^4(a^2-b^2)^2x^2y^2.$$

88. The normals at the points P, Q, R, S on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent. Prove that two parabolas can be drawn through $PQRS$ and the angle between their axes is $2\tan^{-1}\frac{b}{a}$.

89. PQ, PR are focal chords of an ellipse; show that the tangents at Q, R intersect on the normal at P.

90. Show that the diameter of the circle which touches the ellipse at any point and passes through the extremities of a diameter of the ellipse has the constant ratio $a^2+b^2:2ab$ to the conjugate diameter of the point.

91. If PQ is a focal chord of an ellipse and R is the intersection of the tangent at P and the normal at Q, show that QR is bisected by the minor axis.

92. From the foot of the perpendicular let fall from the centre of an ellipse on the tangent at a point P, another tangent is drawn to touch the ellipse in Q. Show that the eccentric angles of P, Q (θ, ϕ) are connected by $b^2\tan\frac{1}{2}(\theta+\phi)=a^2\tan\theta$, and the other extremity of the diameter through Q lies on the normal at P.

93. Points P, Q, one on each of the central conics $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\left(\frac{1}{c^2}-\frac{1}{b^2}\right)x^2+\left(\frac{1}{c^2}-\frac{1}{a^2}\right)y^2=1$, subtend a right angle at the common centre. Prove that PQ touches the circle $x^2+y^2=c^2$.

94. The normals at the extremities of two chords of an ellipse are concurrent. If one chord becomes a tangent to the given ellipse, the other is normal to $b^{-2}x^2+a^{-2}y^2=a^2b^2(a^2-b^2)^{-2}$.

95. The normals at the points P, Q, P', Q' of an ellipse are concurrent. If PQ passes through a fixed point, find the locus of the middle point of $P'Q'$.

96. If the circles of curvature at two points on an ellipse intersect on the curve, show that their radical axis is parallel to the chords joining the extremities of diameters conjugate to those through the given points.

97. From a fixed point E on a central conic chords are drawn equally inclined to the axes and cutting the curve again at P and Q. Find the locus of the centroid of the triangle PQE.

98. From any point on the normal at the point whose eccentric angle is α, two other normals are drawn to an ellipse. Prove that the locus of the point of intersection of the corresponding tangents is the hyperbola $$bx \sin\alpha + ay\cos\alpha + xy = 0.$$

99. If a circle be described on any focal chord of an ellipse as diameter, the line joining the other points of intersection of the circle and ellipse passes through a fixed point.

100. If the sum of the angles made by the four normals from (x, y) to an ellipse with the axis of x be an odd multiple of a right angle, then the locus of (x, y) is $$x^2 - y^2 = a^2 - b^2.$$

101. Prove that the cosine of the angle which the tangent at any point on an ellipse makes with the line joining the point to a focus bears a constant ratio to the cosine of the angle which the tangent makes with the axis major.

102. Show that the tangents of the inclinations to the major axis of the four normals that can be drawn from the point (x, y) to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ are given by the equation
$$(a^2 + b^2m^2)(y - mx)^2 = (a^2 - b^2)^2 m^2.$$

103. If (x_1, y_1) and (x_2, y_2) are two points on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ the tangents at which meet in (x, y) and the normals in (ξ, η), prove that $$a^2\xi = e^2 x x_1 x_2;\ b^2\eta = e^2 y y_1 y_2.$$

104. If the circle of curvature at two points P and Q meet in L and M, of which L is on the ellipse, prove that LM meets the curve in the same point as the diameter which bisects PQ.

105. P, Q, R are points on an ellipse such that PQ and PR are focal chords; prove that the equation of QR is $\dfrac{x\xi}{a^2} + \dfrac{y\eta}{b^2}\,\dfrac{1+e^2}{1-e^2} + 1 = 0$, where P is the point (ξ, η).

106. A chord PQ of a conic passes through a fixed point. If the circle on PQ as diameter meets the conic again in P' Q', show that P' Q' also passes through a fixed point.

107. The normals to an ellipse at P, Q, R meet in a point, and also the sum of the eccentric angles at these points is constant. Show that the locus of their intersections is a straight line.

108. The circles of curvature at the points L, M, N on an ellipse meet the ellipse at the same point O, whose eccentric angle is θ; find the eccentric angles of L, M, N, and show that the circle LMN passes through O.

109. If A and B be the angles subtended at any point on an ellipse by two conjugate diameters, then $\cot^2 A + \cot^2 B = \dfrac{(a^2-b^2)^2}{4a^2b^2}$.

110. If T be the intersection of tangents to an ellipse at the extremities of a chord PQ normal at P, prove that the perpendicular from T on the diameter through P intercepts on PQ a length PN equal to the radius of curvature at P.

111. Prove that if the circles of curvature at three points P, Q, R on an ellipse pass through the same point on the ellipse, the centroid of the triangle PQR lies on one of the axes of the ellipse.

112. Normals are drawn to an ellipse from any point on its evolute. Find the locus of the centre of the circle through the three points of incidence.

113. The normal at any point P of an ellipse meets the major axis in G; a point Q is taken in the tangent at P so that $PQ = k\,PG$; show that the locus of Q is a concentric ellipse.

114. Normals are drawn to an ellipse from any point on either of the axes; show that all conics through the feet of these normals pass through the extremities of either the major or minor axis.

115. Find the locus of the middle points of a chord of an ellipse which subtends a right angle at the centre.

116. Prove that the envelope of a chord of an ellipse whose middle point lies on a fixed line is a parabola.

117. Find the radius of curvature at a given point of the hyperbola $x^2 - y^2 = a^2$. Also find the locus of the intersection of the chord of curvature and the diameter parallel to the tangent at a point.

118. Find the locus of the vertices of an equilateral triangle circumscribed to an ellipse.

119. From any point on a fixed normal to an ellipse the remaining three normals are drawn. Prove that the lines joining the feet of these normals touch a fixed parabola, and that the tangents to the ellipse at these points form a triangle self-conjugate with respect to this parabola.

120. A point moves so that the product of the lengths of the tangents from it to an ellipse is in a constant ratio to the product of its distances from the foci. Show that it lies on a similar ellipse.

121. PP' is a double ordinate of an ellipse and the normal at P meets CP' in Q. Show that the locus of a point which divides PQ in a given ratio is an ellipse.

122. A system of circles have their centres on a fixed circle of radius r', their radii being constant and equal to r. Find the locus of points on these circles the tangents at which are in a given direction.

123. If O is the centre of curvature at the point P of an ellipse, and R be taken on OP so that $OP = PR$, show that P and R are conjugate points with respect to the director circle.

124. A, B, C are the vertices of a triangle of maximum area inscribed in an ellipse; P, Q, R the centres of curvature corresponding to A, B, C. Find the locus of the centroid of the triangle PQR.

125. Show that the locus of the second point of intersection of two circles described on two conjugate semi-diameters of an ellipse as diameters is the inverse of a concentric ellipse with regard to a circle whose centre is at the centre of the ellipse.

126. SY is the perpendicular from the focus on a tangent to an ellipse at P, and SR is parallel to the tangent at P and meets the diameter through Y in R. Prove that $SYPR$ is a rectangle.

127. P_1, P_2, P_3, P_4 are four points on an ellipse taken in this order, and $P_1 P_3$, $P_2 P_4$ are the ends of pairs of conjugate diameters. Show that if the anharmonic ratio of the pencil determined at any point of the conic by $P_1 P_2$ and $P_3 P_4$ is constant, the envelope of $P_1 P_2$ is a similar and similarly situated ellipse.

128. Perpendicular tangents to an ellipse intersect on the director circle and their chords of contact touch a coaxal ellipse whose semi-axes are third proportionals to the radius of the director circle and the semi-axes of the given ellipse.

129. Perpendiculars SM, HN are drawn to the focal distances SP, HP of any point P on an ellipse, and meet the tangent at P in M and N. Prove that if the eccentricity of the ellipse is not less than $\frac{1}{\sqrt{2}}$ the minimum value of $PM \cdot PN$ is $4b^2$, but that otherwise it is

$$b^2 e^{-2}(1-e^2)^{-1}.$$

130. Prove that the chord joining the two points on $x^2 - y^2 = a^2$ whose abscissae are $a\cosh\alpha$, $a\cosh\beta$ is the line

$$x\cosh\tfrac{1}{2}(\alpha+\beta) - y\sinh\tfrac{1}{2}(\alpha+\beta) = a\cosh\tfrac{1}{2}(\alpha-\beta).$$

AA' are the vertices of $x^2-y^2=a^2$, and P, Q the points whose parameters are $\alpha+\delta$, $\alpha-\delta$, where δ is constant. Prove that the locus of the intersection of AP, $A'Q$ is a rectangular hyperbola with parallel asymptotes.

131. The circle of curvature at any point P of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ meets the curve at Q; if $(a\sec\theta,\, b\tan\theta)$ and $(a\sec\phi,\, b\tan\phi)$ be the coordinates of P, Q respectively, find the relation connecting θ and ϕ. Determine also the locus of the pole of the chord PQ with respect to the hyperbola.

132. From O, the centre of curvature at any point on an ellipse, the other normals OQ, OR are drawn to the ellipse; find the locus of the middle point of QR.

133. The locus of the extremities of equi-conjugate diameters of ellipses whose principal axes lie along the coordinate axes and each of which touches the curve $x^4+y^4=c^4$ at four distinct points is

$$c^4(x^4-y^4)=x^4y^4.$$

134. Show that the normals drawn to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ from any point on the curve $(c^4-a^2x^2-b^2y^2)^3=54a^2b^2c^4x^2y^2$, $(c^2\equiv a^2-b^2)$ form an harmonic pencil.

135. If Y is the foot of the perpendicular drawn from the centre of a given ellipse upon the tangent to the curve at a point P, find the position of P when the intercept PY is equal to the corresponding intercept on the tangent at a given point on the ellipse. Find the number of solutions.

136. If a chord of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ touches the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, show that the locus of its middle point is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2}{a^2}-\frac{y^2}{b^2}.$$

137. The sum of the squares of the tangents from any point O on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=p+1$ to the ellipse $\dfrac{x^2}{a^2}\quad\dfrac{y^2}{b^2}=1$ is proportional to the square of the tangent from O to the circle

$$x^2+y^2+\frac{p+1}{p-1}\cdot(a^2+b^2)=0.$$

When $p=1$ show that the sum is constant and equal to the square on the radius of the director circle.

138. Show that the locus of the intersections of the chords of intersection of the ellipse and the osculating circles at the extremities of two conjugate semi-diameters is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2\left(\dfrac{xy}{ab}\right)^{\frac{2}{3}}$.

139. If $(x_1, y_1)\,(x_2, y_2)\,(x_3, y_3)$ are the vertices of a triangle self-conjugate with respect to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, the area of the triangle is $\dfrac{(x_1 y_2-x_2 y_1)\,(x_2 y_3-x_3 y_2)\,(x_3 y_1-x_1 y_3)}{2a^2b^2}$.

140. From any point on the circle $x^2+y^2+2gx+a^2-b^2=0$ two tangents are drawn to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$; if they subtend angles α and β at the foci respectively, show that the ratio of the sines of these angles is constant. For what value of g is the ratio one of equality?

141. From either of the points of intersection of the curves

$$(x^2+y^2)^2=2a^2e^2(x^2-y^2)\,;\ \text{ and }\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=2,$$

four normals are drawn to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$; show that the product of these normals is a^2b^2.

142. A circle is drawn to meet an ellipse at four points. Show that the sum of the cotangents of the angles between the curves at these points is zero.

143. From a point T tangents TP, TQ are drawn to an ellipse. If the area of $SPTQ$ be constant find the locus of T.

144. PP' is a diameter of an ellipse, O the centre of curvature at P, and M and N the feet of the other normals which can be drawn from O to the ellipse. Prove that the points P, P', M, N lie on a circle and find the equation of the locus of the centre of this circle.

145. The normal at P to the rectangular hyperbola $xy=c^2$ meets the curve at Q. If x, y are the coordinates of P and ξ, η those of Q, prove $\xi x^3=\eta y^3=-c^4$.

146. From a point P are drawn two tangents to a rectangular hyperbola. The tangents of their inclinations to an asymptote being p, q, show that if the ratio of $l-pq$ to $(\sqrt{p}-\sqrt{q})^2$ is constant, the locus of P is another rectangular hyperbola.

147. Find a point on a given hyperbola such that the straight line joining the feet of the perpendiculars from it to two given diameters shall be a minimum.

148. A parabola has 4-point contact with a rectangular hyperbola. Prove that the directrix of the parabola bisects at right angles the central radius vector of the hyperbola drawn to the point of contact.

149. Find the locus of the middle points of chords of a rectangular hyperbola of constant length.

150. The locus of the middle points of all chords of an hyperbola, which pass through a fixed point, is another hyperbola having the same asymptotes.

151. Equal parabolas are drawn having the asymptotes of a rectangular hyperbola as axis and tangent at the vertex respectively. Show that the straight lines joining their intersections with the hyperbola are parallel to the axes of the hyperbola.

152. Find the locus of the intersection of tangents to a rectangular hyperbola at the extremities of chords: (i) of constant length, (ii) which pass through a fixed point.

153. The normal and chord of curvature at any point on a rectangular hyperbola make equal angles with the asymptotes.

154. The chords of curvature at four points on an hyperbola are concurrent. Show that the four points are concyclic.

155. Find the conditions that the chords of curvature at the points of intersection of $Ax+By = C$ and $A'x+B'y = C'$ with an hyperbola should be concurrent.

156. The tangents at the ends of a focal chord of a rectangular hyperbola intersect at T and the normals at N. Show that TN passes through the other focus.

157. A, B, C are three points on an hyperbola and D is the centroid of the triangle ABC. If the ellipse circumscribing ABC and having its centre at D has a pair of conjugate diameters parallel to the asymptotes of the hyperbola, show that D is on the hyperbola, and that the other extremity of the diameter through D is the fourth intersection of the ellipse and hyperbola.

158. Show that the mean distance of the points of contact of the four rectangular hyperbolas which can be drawn from any point to osculate a parabola, is equal to the distance of the given point, all these distances being measured from the axis of the parabola.

159. The sum of the reciprocals of the distances from either asymptote of the points of contact of the four parabolas which can be drawn through any given point to have 4-point contact with an hyperbola, is four times the reciprocal of the distance of the given point from the same asymptote.

160. Two chords AB, CD of a rectangular hyperbola make angles θ, ϕ with the transverse axis. Show that if ψ is the angle between the axes of the parabolas drawn through $ABCD$, then

$$\cos\psi = \sin(\theta+\phi)\sec(\theta-\phi).$$

161. Through a fixed point on a rectangular hyperbola is drawn a circle such that the chords joining the three other points of intersection to the fixed point are equally inclined to one another. Show that the centre of the circle is the point on the hyperbola diametrically opposite to the fixed point.

162. Triangles are inscribed in $x^2+y^2 = 2a(x+y)$, and their sides touch the R. H. $xy = a^2$. Prove that their centroids lie on

$$3(x^2+y^2) = 2a(x+y).$$

163. To a rectangular hyperbola with centre C and focus S normals are drawn from a point P. Show that if these normals make angles θ_1, θ_2, θ_3, θ_4 with one of the asymptotes

$$\Sigma \operatorname{cosec} 2\theta = \frac{CP^2}{CS^2}.$$

164. A rectangular hyperbola passes through the centre of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, and meets it in four real points. If t_1, t_2, t_3, t_4 are the tangents of half the eccentric angles of the points of intersection, show that

$$(1+t_1t_2t_3t_4)(a^2+2b^2)+(t_1t_2+t_1t_3+t_1t_4)a^2 = 0.$$

165. Prove that the coordinates of the centre of perpendiculars of the triangle whose vertices are

$$(a+b\mu_1^{-1},\ a\mu_1+b)\ (a+b\mu_2^{-1},\ a\mu_2+b)\ (a+b\mu_3^{-1},\ a\mu_3+b)$$

may be written $\quad (a+b\mu^{-1},\ a\mu+b),$

where $\quad \mu^{-1} = -a^2b^{-2}\mu_1\mu_2\mu_3,$

and interpret this result geometrically.

166. The locus of the middle points of all chords of an hyperbola which cut off a constant intercept from an asymptote is a straight line parallel to the other asymptote.

167. Prove that if at a point on an hyperbola the sum of the tangents of the angles which the normal makes with the asymptotes is 2, then the vertex of the parabola of closest contact at the point lies on a line through the point inclined at an angle $\tan^{-1}3$ to the normal, and at a distance from the normal equal to three-eighths of the radius of curvature at the point.

168. Show that the parabola which has contact of the third order at any point on a rectangular hyperbola cuts each asymptote in points equidistant from the common tangent.

169. The tangent at any point P of the cissoid $y^2(a-x) = x^3$ cuts the curve again at Q, and R is a point on PQ such that $PR = 2RQ$. Show that if the straight lines joining R, P, Q to the origin make angles θ, α, β respectively with the axis of x, then $\cot\theta = \tan\alpha - \cot\beta$.

170. A nodal cubic intersects in points P, P' two lines which are harmonic conjugates to the tangents at the node. Prove that the tangents at PP' meet on the curve.

171. Show that the locus of the intersection of tangents to the curve $x^3 = ay^2$ which are at right angles is a parabola.

172. A straight line meets the cubic $a(x^2+y^2) = x^3$ at P, Q, R. Show that the sum of the inclinations of the lines joining P, Q, R to the conjugate point to any fixed straight line is constant.

173. Three tangents are drawn from any point to the cubic

$$27a(x-y+a)^2 = 4x^3,$$

whose inclinations to the x-axis are α, β, γ.

Prove that $\tan\alpha + \tan\beta + \tan\gamma = 3.$

If $\alpha+\beta+\gamma = 180°$, show that the point of intersection of the tangents lies on the straight line $y+a = 0$.

174. A variable straight line meets a nodal cubic in three points A, B, C. The chord AB subtends a right angle at the node. If the tangents at the node are perpendicular, the point C is fixed.

175. The tangent at any point P to the curve $y^2(a-x) = x^3$ cuts the curve again in Q. Show that if p_1, p_2 are the perpendiculars on the asymptotes from P and Q $4p_2^{-1} - p_1^{-1} = 3a^{-1}$.

176. From any point (x, y) three tangents are drawn to a semi-cubical parabola; the circle through their points of contact meets the cubic again in three other points. Prove that the tangents at these points meet at some point (X, Y), where $3x+3X+a = 0$.

177. The sum of the inclinations to the x-axis of the radii vectores from the origin to any six points on the cubic $a(x^2+y^2) = x^3$ which lie on a conic is a multiple of two right angles.

178. Any three straight lines parallel to the axis of x meet the cubic $ax^2 = y^3$ in six points lying on a conic.

179. The tangent at any point A on the curve $(x+a)^3-3a^2y=0$ meets the curve again at A_1, the tangent at A_1 meets the curve again at A_2, and so on. Show that the locus of the intersection of the tangents at A_{2m}, A_{2n}, where m and n are constant, is a cubic with the same cuspidal point as the given cubic.

180. If any straight line through the point (a, a) meets the cubic $(x-a)^3 = y^2(x-y)$ in the points A and B, prove that the tangents at A and B meet on the axis of y.

181. The tangent at any point on the curve $(x+y)(x-y)^2 = a^3$ makes an angle θ with the x-axis, and the radius vector to the point from the origin an angle ϕ. Show that $3\cos(\theta+\phi)+\sin(\phi-\theta) = 0$.

182. The normals to the curve $a(x-y)^2 = x^3$ at five points are concurrent. Prove that the sum of the abscissae of these points is $\dfrac{11a}{3}$.

183. The tangents at two points (x_1, y_1) (x_2, y_2) on the cubic $a(x-y)^2 = x^3$ are parallel. Prove that $3y_1y_2+x_1x_2 = 0$.

184. Three tangents are drawn to a cuspidal cubic from any point and meet the curve again in three other points. Show that the tangents at these three points are concurrent.

Show also that if the point from which the tangents are drawn lies on a fixed line, the intersection of the latter three tangents also lies on a fixed line.

185. The tangent at any point P on the curve $ay^2 = x^3$ meets the curve again at Q, the tangent at Q meets the curve again at R, and the tangent at R cuts PQ in S. Show that the locus of S is $343\,ay^2 = 100\,x^3$.

186. The tangent at P to a cuspidal cubic meets the curve again at Q. Show that the locus of the point which divides PQ in a fixed ratio is a cuspidal cubic.

187. Find the locus of the middle point of the chord of contact of tangents from points on the cubical parabola $y^3 = cx^2$ to the hyperbola $xy = 4c^2$, and prove that the hyperbola, the cubical parabola, and the locus meet at a point.

188. A straight line passes through the point $\left(0, \dfrac{a}{3}\right)$ and meets the cubic $y^3 = ax^2$ at three points P, Q, R. A fourth point S is taken so that P, Q, R, S are harmonic conjugates. Find the locus of S.

189. Find the locus of the intersections of corresponding points on a nodal cubic.

190. A straight line cuts the cubic $y^3 = ax^2$ in three points; the lines joining these points to the origin are such that one bisects the angle between the other two. Find the envelope of the straight line.

191. A straight line cuts the evolute of a parabola in three real points, from each of which the normal other than the radius of curvature is drawn. Show that the centres of curvature at the feet of these normals are collinear.

192. Show that the coordinates of any point on the Witch of Agnesi can be represented by $(a\cos^2\theta,\ a\tan\theta)$ and find the tangent at the point θ.

193. Any cubic which passes through four pairs of corresponding points on a nodal cubic passes also through a point of inflexion on the curve.

194. Find the locus of a point which moves so that one of the tangents from it to the cubic $ay^2 = x^3$ bisects the angle between the other two.

195. The sum of the inclinations to the axis of x of the tangents from a point to the cubic $ay^2 = x^3$ is constant. Find the locus of the point.

196. Find the condition that the tangents at three points on the curve $(x+y)(x-y)^2 = a^3$ should be concurrent.

197. The points of contact of tangents from any point on the line $3x-2a = 0$ to the cubic $a(x^2-y^2) = x^3$ lie on a circle through the origin. Show also that the centre of this circle lies on a fixed straight line.

198. If (x_1, y_1) (x_2, y_2) (x_3, y_3) are the coordinates of any three collinear points on the cubic $x^3+y^3=a^3$, show that

$$(x_1+x_2+x_3)(y_1+y_2+y_3) = (x_1y_1+x_2y_2+x_3y_3).$$

199. Find the locus of the intersections of tangents to the curve $y^2(a-x) = x^3$ at two points which are such that the sum of the squares of the cotangents of the angles made by the radii vectores from the origin to them with the axis of x is equal to $\frac{1}{4}$.

200. A variable circle through the origin cuts the cubic $a(x^2-y^2) = x^3$ in four points; if the sum of the angles which the radii vectores from the origin to these points make with the axis of x is constant, find the locus of the centre of the circle.

201. The tangent at any point P on the curve $y^2(a-x) = x^3$ meets the curve again in Q and the tangent at Q meets the curve again in R. If O is the cusp, show that

$$\cot ROQ - \cot POQ = \tfrac{1}{2}\cot POR.$$

202. The tangent at the point $(4x, 4y)$ to the curve $(x+y)^2(x-y)=a^3$ meets the curve again at the point $(7x-9y,\ 7y-9x)$.

203. Show that the tangents at the points of intersection of the straight line $9x-8a=0$ and the curve $a(x^2-y^2)=x^3$ are parallel to the tangents at the node. Show also that the normals at these points meet on the axis of x, at a distance $\frac{16a}{27}$ from the node.

204. Find the equation giving the other points of intersection of the normal at a point P to the curve $a(x^2-y^2)=x^3$; if the normal meets the curve at infinity, show that P is one of two corresponding points.

205. Find the area of the triangle formed by the tangents at the points of intersection of the line $3x-2y-3a=0$ and the cubic $a(x^2-y^2)=x^3$.

206. A circle through the origin O cuts the cubic $ay^2=x^3$ in four points P, Q, R, S. If PQ is the tangent at P to the cubic, prove (1) that the tangents at R and S meet on the cubic $32ay^2+(9x+a)^2(3x+a)=0$, and (2) that if α, β, θ are the inclinations of OR, OS, PQ to the x-axis, then $\tan(\alpha+\beta)\tan\theta=1$.

207. Show that four tangents can be drawn to the curve $a(x^2+y^2)=x^3$ making a given angle θ with the axis of x, and if the radii vectores from the origin to the points of contact make angles $\alpha, \beta, \gamma, \delta$ with the axis of x,

$$3\tan\alpha\tan\beta = 3\tan\gamma\tan\delta = 1,$$

and find the equations of the chords of contact in terms of θ.

208. Find the asymptotes of the curve $(3x-a)^3=27x^2y$, and the coordinates of the finite points of intersection of these asymptotes and the curve.

Find the direction of a straight line through the point $\left(\frac{a}{3}, 0\right)$ such that the intercept made on it by the axes is bisected by the curve.

209. Two radii vectores OP, OQ of the curve $ay^2=x^3$, at right angles to one another, are drawn through the origin. Show that the envelope of PQ is a conic; and that if PQ cuts the curve again in R, and the tangent to the curve from R touches the curve in T, then OP, OQ are the bisectors of the angles between OT and OX.

210. The normals from a point to the cubic $ay^2=x^3$ make angles with the axis of x, whose sum is constant. Find the locus of the point.

211. Show that the ratio of the radii of curvature at points on the curves $xy = a^2$ and $x^3 = 3a^2y$ which have the same abscissae varies as the square root of the ratio of the ordinates.

212. P is any point on a cissoid and Q the point where the tangent at P meets the curve again. Prove that the pencils determined at the cusp by P and Q respectively are homographic.

213. An hyperbola is drawn with its asymptotes parallel to the coordinate axes so as to have three-point contact with the curve $c^2y = x^3$. Prove that the locus of its centre is the curve $4c^2y + x^3 = 0$.

214. From any point on $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ perpendiculars are drawn meeting the coordinate axes in H, K. Show that the envelope of the parabolas drawn to touch the axes at H and K is $x^{\frac{2}{7}} + y^{\frac{2}{7}} = a^{\frac{2}{7}}$.

215. A line drawn from the foot of the ordinate of any point on the ellipse $a^2x^2 + c^2y^2 = a^4$ parallel to the normal at the point touches the curve $a^{-\frac{2}{3}}x^{\frac{2}{3}} + c^{-\frac{2}{3}}y^{\frac{2}{3}} = 1$.

216. Prove that the coordinates of three collinear points on the curve $x^3 + y^3 - 3axy + c = 0$ are connected by the relation

$$x_1x_2x_3 + y_1y_2y_3 + c^3 = 0.$$

217. If the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$, then

$$c = a + b.$$

ANSWERS AND HINTS TO SOLUTION

1. (1) A straight line.

   (2) A conic passing through the origin.

   (3) A conic passing through O, whose asymptotes are parallel to the given straight lines.

2. The straight lines make angles $\alpha \pm 30^\circ$ with the x-axis, hence the roots of the equation $bt^2 - 2ht + a = 0$ are $\tan(\alpha \pm 30^\circ)$.

4. Equivalent to finding the condition that the product of two of the roots of the equation $yt^4 + 2t^3(x+1) + 2t(x-1) - y = 0$ should equal a given constant.

5. Area $= \dfrac{\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}$.

6. $4(a^2n - 2abm + b^2l)\sqrt{m^2 - ln} \div \{(l-n)^2 + 4m^2\}$.

7. $x(x-\alpha)^2 + y(y-\beta)^2 = 0$.

8. $\cos\alpha + \cos\beta + \cos\gamma = 0$.

9. $(x^2+y^2)(ax+by+c)^2 = d^2(ax+by)^2$.

10. $y^2 = (x^2+y^2-4x+2)(x^2+y^2-2x+1)$.

13. A rectangular hyperbola.

14. $\alpha + \beta + \gamma = \dfrac{\pi}{2}$.

15. Let (lk, mk) be the orthocentre and suppose

$$b(y-\lambda x)(y-\mu x) \equiv by^2 + 2hxy + ax^2;$$

then the straight line through the orthocentre perpendicular to $y - \lambda x = 0$, and the straight lines $y - \mu x = 0$, $lx + my = 1$ are concurrent, hence a simple equation for k.

16. If (x_1, y_1) (x_2, y_2) are the ends of a diameter, the circle is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0,$$

and equations giving $x_1 x_2$ and $\overline{x_1 + x_2}$, &c., can be at once formed.

19. $x\cos(\alpha+\beta+\gamma) - y\sin(\alpha+\beta+\gamma) + \cos 2(\alpha+\beta+\gamma) = 0$.

20. If the fixed point is $(0, 0)$ and the fixed straight line $x = d$, then the locus is $\quad 6x - 2\sqrt{3}\,y = d(\sqrt{3} - 1)$.

21. If (h, k) is the fixed point the locus is

$$\frac{2h}{x}+\frac{2k}{y}=1.$$

23. Take the axis of y in the fixed direction, if $x=h$ is one of the parallel chords, the point whose locus is required is $\left(h, \Sigma\frac{y}{n}\right)$, the values of y being given by the equation

$$ay^n-(bh+c)\,y^{n-1}+\ldots=0.$$

Hence $\Sigma\frac{y}{n}=\frac{bh+c}{na}$, therefore the locus is $nay=bx+c$.

24. Let the focus be $(0, 0)$, and the given point $(a, 0)$, the eccentricity is $\sqrt{2}$. General coordinates of the foot of the directrix are $\left(a\cos\theta+\frac{a}{\sqrt{2}}\right)\sin\theta$; $\left(a\cos\theta+\frac{a}{\sqrt{2}}\right)\cos\theta$. If (x, y) is the corresponding point on the inverse of the locus with respect to the origin,

$$\frac{k^2x}{x^2+y^2}=\left(a\cos\theta+\frac{a}{\sqrt{2}}\right)\sin\theta;\quad \frac{k^2y}{x^2+y^2}=\left(a\cos\theta+\frac{a}{\sqrt{2}}\right)\cos\theta.$$

Eliminate θ; then $2\,(ay-k^2)=a\,(x^2-y^2)$, which is a rectangular hyperbola. Conversely the locus is the inverse of this R. H. Since $CS=2\,SX$ the second part of the question follows.

25. The bisectors are parallel to

$$\begin{vmatrix} x^2 & xy & y^2 \\ b & h & a \\ 1 & \cos\omega & 1 \end{vmatrix}=0,$$ and also pass through the point $\left(\frac{G}{C}, \frac{F}{C}\right)$, but $Cx-G=bX-hY$ and $Cy-F=aY-hX$.

26. The condition gives $\Delta=cC$.

27. Vide Chap. II, § 15.

29. $(x-a)^2+y^2=c^2+a^2$, where $2c=$ the intercept.

30. Question reduces to showing that $\left(\frac{(a-b)^2}{2h}+2h\right)\sin 2\theta$ is positive.

31. Let the straight lines be $y=0$, $y=x\tan\theta$, $y=x\tan\theta'$, it is required to prove that

$$\left(\frac{1}{OB}+\frac{1}{OB'}\right)\sin\theta'=\left(\frac{1}{OC}+\frac{1}{OC'}\right)\sin\theta+\left(\frac{1}{OA}+\frac{1}{OA'}\right)\sin(\theta'-\theta).$$

33. a^2-8b^2 is positive.

36. $x^2-y^2-6ax=a^2$.

37. $y^2=2a\,(x-a)$.

38. If $\cot^{-1} c$, $\cot^{-1} d$ are the given directions the locus is

$$(x-cy+ac^2)(c^2+1) = (x-dy+ad^2)(d^2+1).$$

39. If (h, a) is the fixed point the envelope is

$$x(x^2+y^2-2hx+h^2-4ah)+ay^2 = 0\,;$$

when $h = a$ this reduces to $(x+a)(x^2+y^2-3ax) = 0$.

42. $4a(x-8a)(5y^2-x^2+6ax-9a^2) = (y^2+x^2-10ax+21a^2)^2$.

46. If P is the point λ, the ordinate of G is $-a\lambda$, therefore G is the point $(a\lambda^2+3a,\ -a\lambda)$.

49. $9y^2 = 4a(x+4y\cot\alpha-2a\cot^2\alpha)$.

51. $y^2+4(x+2a)^2 = 16r^2$.

52. $2\sqrt{7}\,a$.

57. Locus is $y^2 = 4a(x-a)$.

If μ_1, μ_2, μ_3 are the parameters of the second points, the circle condition gives $\Sigma\mu\,\Sigma\dfrac{1}{\mu} = 1$, and these parameters are connected with those of the feet of the normals by the relation $\mu = -\dfrac{\lambda^2+2}{\lambda}$, where $a\lambda^3+a\lambda(2a-x)-y = 0$, P being the point (x, y).

60. The equation of the circle may be easily obtained by expressing that the angle subtended by the chord of contact at (x, y) equals the angle between the tangents. The locus is $2ay^2 = (a-c)^2(x-a)$.

61. $x^2+y^2+5{\cdot}36x-1{\cdot}28y+10{\cdot}92 = 0$.

63. $x^2\tan^2\alpha+y^2\sec^2\alpha = 2ax(1+\sec^2\alpha)-a^2\tan^2\alpha$.

65. Vide Ex. 60.

67. The orthocentre lies on the directrix, hence if $\lambda_1, \lambda_2, \lambda_3$ are the parameters of the points of contact of the sides, the point of intersection of the directrix and one perpendicular of the triangle gives $\Sigma\lambda+\lambda_1\lambda_2\lambda_3 = \text{a constant} = \dfrac{c}{a}\cdot$ The radius of the circumscribing circle is given, hence $(1+\lambda_1^2)(1+\lambda_2^2)(1+\lambda_3^2) = \text{constant} = \dfrac{k}{a}\cdot$ For the locus

$$3x = a\Sigma\lambda_1\lambda_2\,;\quad 3y = 2a\,\Sigma\lambda.$$

Result: $\quad 27y^2+9x^2-6ax-9cy = ka-c^2-4a^2$.

68. $x^2+4y^2-3ax+2a^2 = 0$.

69. The parameters of the points are given by the equation

$$4\lambda^6-35\lambda^4-21\lambda^3+49\lambda^2+21\lambda-18 = 0,$$

whose roots are $-1,\ +1,\ 3,\ \frac{1}{2},\ -2,\ -\frac{3}{2}$.

But $1+\frac{1}{2}-\frac{3}{2} = 0$ and $3-2-1 = 0$, therefore the normals at $1, \frac{1}{2}, -\frac{3}{2}$, and $3, -2, -1$ respectively meet in a point.

71. Radius $= \dfrac{QR}{2\sin QPR}$.

74. The equation of the polar of the focus with respect to the R.H. having 4-point contact at the point λ is divisible by $1+\lambda^2$.

76. (i) The point $(-2a, 0)$. (ii) $y^2 = 2a(x+2a)$.

80. $4a$.

81. If λ, μ are the parameters of the ends of one chord of the system, the circle is

$$(x-a\lambda^2)(x-a\mu^2)+(y-2a\lambda)(y-2a\mu) = 0, \text{ and } \lambda+\mu = 2\cot\alpha.$$

Result: $$(x+4a)^2+8a\cot\alpha\,(x\cot\alpha+y) = 0.$$

84. $(x^2+y^2-a^2-b^2)^2\tan^2\alpha = 4a^2b^2\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1\right)$.

85. i.e. the locus of the intersection of the polar of $(a\cos\theta, a\sin\theta)$ and the diameter $$y = x\tan\theta.$$

86. If P is the point θ, Q is -3θ.

Hence PQ is $$\frac{x}{a}\cos\theta-\frac{y}{b}\sin\theta-\cos 2\theta = 0,$$

therefore RR' is $$\frac{x}{a}\sec\theta-\frac{y}{b}\operatorname{cosec}\theta+\sec 2\theta = 0,$$

and locus required is that of the pole of the latter straight line.

87. Locus is that of the intersection of the normal and the diameter conjugate to it.

88. The parabolas are of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1 = k\left(\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-c\right)\left(\frac{x}{a}\sec\theta+\frac{y}{b}\operatorname{cosec}\theta+\frac{1}{c}\right).$$

90. The circle is of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1 = k\left(\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-1\right)\left(\frac{x}{a}\cos\theta-\frac{y}{b}\sin\theta\right).$$

91. The condition that the x-coordinate of R should be equal and opposite to that of Q is the same as that QR should be a focal chord.

95. If $P'Q'$ is $$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-c = 0,$$

then PQ is $$\frac{x}{a}\sec\theta+\frac{y}{b}\operatorname{cosec}\theta+\frac{1}{c} = 0.$$

But PQ passes through (x', y') and the locus is that of the intersection of $P'Q'$ and the diameter conjugate to it.

Result: $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{x'}{x}+\frac{y'}{y}\right) = -1.$$

96. The difference of the eccentric angles of the two points is $\frac{2\pi}{3}$, and the radical axis is perpendicular to the line of centres.

97. $\left(\frac{3x}{a} - \cos\alpha\right)^2 + \left(\frac{3y}{b} - \sin\alpha\right)^2 = 4\cos^2\alpha.$

98. The chord of contact of the tangents is

$$\frac{x}{a}\sec\frac{\theta+\alpha}{2} + \frac{y}{b}\operatorname{cosec}\frac{\theta+\alpha}{2} + \sec\frac{\theta-\alpha}{2} = 0.$$

99. $\left(\frac{a^2+b^2}{ae},\ 0\right).$

101. The ratio is $\frac{1}{2}e$.

103. If the points are θ, ϕ, obtain the values of (x, y) and (ξ, η) by cross-multiplication from the equations of the tangents and normals, and divide (without previous simplification) the corresponding results.

106. If one fixed point is (x', y'), the other is

$$\left(\frac{a^2+b^2}{a^2-b^2}\,x';\quad \frac{b^2+a^2}{b^2-a^2}\,y'\right).$$

107. The eccentric angle of the foot of the fourth normal from the point is constant, hence the locus is the normal at this point.

108. $\frac{-\alpha}{3}$, $\frac{2\pi-\alpha}{3}$, $\frac{4\pi-\alpha}{3}$, and OL, MN are equally inclined to the axis.

110. Take the equation of the normal

$$\frac{x-a\cos\theta}{b\cos\theta} = \frac{y-b\sin\theta}{a\sin\theta} = \frac{r}{CD},$$

and the radius of curvature is $\frac{CD^3}{ab}$.

112. If the point on the evolute corresponds to θ, the chord joining the feet of two normals from the point is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta - 1 = 0,$$

hence the chord joining the feet of the other two is

$$\frac{x}{a}\sec\theta + \frac{y}{b}\operatorname{cosec}\theta + 1 = 0;$$

the other common chord of the ellipse and circle is, consequently,

$$\frac{x}{a}\sec\theta - \frac{y}{b}\operatorname{cosec}\theta = 0.$$

Result: $\quad 4\,(a^2x^2+b^2y^2)^3 = (a^2-b^2)^2\,a^2b^2x^2y^2.$

113. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{a^2+\lambda^2b^2}{a^2}$.

115. $\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)^2(a^2+b^2)=\left(\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}\right)a^2b^2$.

117. $4x^2y^2(y^2-x^2)=a^2(x^2+y^2)^2$.

118. $4(b^2x^2+a^2y^2-a^2b^2)=3(x^2+y^2-a^2-b^2)^2$.

121. $\dfrac{x^2}{(a+bk)^2}+\dfrac{y^2}{(b+ak)^2}=1$.

122. A circle whose radius is r'.

124. $16(a^2x^2+b^2y^2)=(a^2-b^2)^2$.

125. $\dfrac{2(x^2+y^2)^2}{a^2b^2}=\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}$.

127. If P is θ, and Q is ϕ, then $(\theta-\phi)$ is constant.

129. $\dfrac{1}{PM\,.\,PN}=\dfrac{1}{4b^2}-\left(\dfrac{1}{2b}-\dfrac{b}{CD^2}\right)^2$, hence $PM\,.\,PN$ is a min. when $CD^2=2b^2$; this is only possible when $a^2>2b^2$. Again,

$$PM\,.\,PN=\frac{a^2}{e^2}\left[1+\cos^2\theta\,(1-e^2\cos^2\theta)^2\right],$$

which gives $\dfrac{a^2}{e^2}$ for a min. when $\theta=\dfrac{\pi}{2}$, and $4b^2$ is less than $\dfrac{a^2}{e^2}$, since the greatest value of $e^2(1-e^2)$ is $\frac{1}{4}$.

131. $3\tan\dfrac{\theta}{2}\left(1+\tan\dfrac{\theta}{2}\tan\dfrac{\phi}{2}\right)+\tan\dfrac{\phi}{2}+\tan^3\dfrac{\theta}{2}=0$.

132. QR is of the form $\dfrac{x}{a}\sec\theta+\dfrac{y}{b}\operatorname{cosec}\theta-1=0$, and the locus is that of the intersections of QR and the diameter conjugate to it.

133. If the curves touch they do so at four symmetrical points, say $(\pm a\cos\theta,\ \pm b\sin\theta)$. Also $a^4\cos^4\theta+b^4\sin^4\theta=c^4$ has equal roots, hence $(a^4+b^4)(b^4-c^4)=b^8$; but the ends of equiconjugate diameters are $\pm\dfrac{a}{\sqrt{2}},\ \pm\dfrac{b}{\sqrt{2}}$, &c.

134. The eccentric angles of the feet of the normals from (x, y) to the curve are given by $(t\equiv\tan\theta)$

$$a^2x^2t^4-2abxyt^3+(a^2x^2+b^2y^2-c^4)t^2-2abxyt+b^2y^2=0.$$

Now $$t_3t_4(t_1+t_2)+t_1t_2(t_3+t_4)=\frac{2by}{ax},$$

and $$(t_1+t_2)+(t_3+t_4)=\frac{2by}{ax}.$$

Hence $$2(t_1t_2+t_3t_4) = (t_1+t_2)(t_3+t_4)$$
by condition for harmonic pencil,
$$= -\frac{4b^2y^2(1-t_1t_2)(1-t_3t_4)}{a^2x^2(t_1t_2-t_3t_4)^2}.$$

But $$3(t_1t_2+t_3t_4) = s_2 = \frac{a^2x^2+b^2y^2-c^4}{a^2x^2} = \frac{3p}{a^2x^2} \text{ (say)},$$

and $$s_4 = \frac{b^2y^2}{a^2x^2}.$$

Therefore $$p(p^2-4a^2b^2x^2y^2)+2a^2b^2x^2y^2(a^2x^2+b^2y^2-p) = 0,$$
i.e. $$p^3+2a^2b^2c^4x^2y^2 = 0,$$
or the point (x, y) must lie on the given locus.

135. If P is the point θ, and α the given point, the condition gives
$$\tan^2\theta \tan^2\alpha = 1-e^2.$$

140. Ratio $= \dfrac{ae+g}{ae-g}$.

144. $4(a^2x^2+b^2y^2)^3 = (a^2-b^2)^2 a^2b^2x^2y^2$.

146. $x^2-y^2 = 4k(xy-c^2)$.

148. Take $x\cos\alpha+y\sin\alpha = p$ for directrix, (h, k) for focus. Substitute $x = c\lambda$, $y = \dfrac{c}{\lambda}$ in the equation of the parabola. This equation has four equal roots, hence $\cot\alpha = \lambda^2$ and $p = \frac{1}{2}c\operatorname{cosec}\alpha$.

149. $(x^2+y^2)(xy-c^2) = kxy$.

150. $xy = 2(kx+hy)$.

152. (i) $c^2(c^2-xy)(x^2+y^2) = x^2y^2d^2$.
(ii) $kx+hy = 2c^2$ (the polar of the fixed point).

155. $AA' = BB'$ and $CC'+c^2(AB'+A'B) = 0$.

156. Vide Ex. 103.

157. The ellipse is of the form $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2} = 1$, where D is the point (h, k).

162. Take $\lambda_1, \lambda_2, \lambda_3$ as the parameters of the points of contact, then $s_1 = \Sigma\lambda$, $s_2 = \Sigma\lambda_1\lambda_2$, $s_3 = \lambda_1\lambda_2\lambda_3$; the vertices of the triangle have coordinates $\dfrac{2as_3}{\lambda(s_1-\lambda)}, \dfrac{2a}{s_1-\lambda}$; the condition that this point should lie on the circle gives an equation whose roots are $\lambda_1, \lambda_2, \lambda_3$, hence, &c.

163. Use $\frac{1}{2}\operatorname{cosec} 2\theta = \cot\theta+\tan\theta = \lambda^2+\dfrac{1}{\lambda^2}$.

165. The points lie on the R. H. $(x-a)(y-b) = ab$. Transfer the origin of coordinates to the point (a, b).

179. If A is the point λ, A_{2n} is the point $4^n\lambda$.

183. If λ, μ are the parameters of the points of contact, $\lambda\mu = -\frac{1}{8}$.

187. $x^3 = 4cy^2$.

188. $(x^2+y^2)(2x^2-3y^2+ay) = x^2(5y-a)^2$.

189. A straight line.

190. $(x^2+y^2)(x^2+3y^2+2ay) = x^2(y+a)^2$.

194. $2ay^2 = \left(x-\dfrac{4a}{9}\right)^2\left(x+\dfrac{2a}{9}\right)$.

195. $27x+4 = 27y\cot\alpha$.

196. $x_1+x_2+x_3 = y_1+y_2+y_3$.

197. If the point is $\left(\dfrac{2a}{3}, k\right)$ the circle is $x^2+y^2+2ky-\dfrac{4ax}{3} = 0$.

198. Let the straight line be $y = mx+c$. The equation giving the x-coordinates of the points of intersection of this line and the curve is

$$(mx+c)^3+x^3 = a^3.$$

Hence $$c\Sigma x+m\Sigma x_1 x_2 = 0.$$

But since $y_1 = mx_1+c$, &c., $\Sigma x_1(y_2+y_3) = 2c\Sigma x+2m\Sigma x_1 x_2$, &c.

200. If α is the sum of the angles the locus is $y-x\tan\alpha = 0$.

205. $\dfrac{1445a^2}{256}$.

207. $x(2\tan^2\theta+3)\pm 3y\tan\theta = 2a(\tan^2\theta+2)$.

208. $x-y-a = 0$, $\left(\dfrac{a}{9}, -\dfrac{8a}{9}\right)$ and $x = 0$. $45°$.

209. $y^2 = 4x(x-a)$.

210. $y\cot\alpha = x+4a$.

213. If (ξ, η) is the centre, the hyperbola is $(x-\xi)(y-\eta) = d^2$; hence $(c\lambda-\xi)(c\lambda^3-\eta) = d^2$ has three equal roots.

214. The equation of the parabola is

$$(x\sec^3\theta+y\operatorname{cosec}^3\theta-a)^2 = 4xy\sec^3\theta\operatorname{cosec}^3\theta.$$

The point $(a\cos^7\theta, a\sin^7\theta)$ is on this parabola and the tangent to the parabola at this point is $x\sec^5\theta+y\operatorname{cosec}^5\theta = a$, which is also a tangent at the same point to the curve $x^{\frac{2}{7}}+y^{\frac{2}{7}} = a^{\frac{2}{7}}$.

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